A

DESCRIPTION

OF THE

VISIBLE NUMERATOR,

WITH

INSTRUCTIONS FOR ITS USE;

Illustrated with Plates.

DESIGNED TO IMPART TO LEARNERS A CLEAR AND AN ADEQUATE KNOWLEDGE OF THE PRINCIPLES OF ARITHMETIC, AND TO ACCOMPANY THE APPARATUS.

By OLIVER A. SHAW.

BOSTON:

T. R. MARVIN, PRINTER, 24, CONGRESS STREET.

1832.

THE VISIBLE NUMERATOR.

THIS apparatus is designed to illustrate to the *Eye*, and thus to convey to the *Mind*, a clear and an adequate idea of the general principles of Arithmetic. It is, in a word, designed to take the same place in *Arithmetic*, which the *Diagrams* occupy in *Geometry*. Its nature and use are founded in the soundest principles of Mental Philosophy, as advanced and maintained by Locke, Reid, Stewart, Brown, and others.

Consisting of a series of solids, so arranged as to impart with rapidity a satisfactory knowledge of a most important branch of Education, this simple apparatus will, in the opinion of many of the most distinguished Mathematical Professors in our country, who have examined it, prove an essential aid in the Art of Teaching, by communicating in a few hours, as much real science, as it has usually cost months, and even years, fully to acquire.

Each set of the Visible Numerator is accompanied by ample directions for its use, and is adapted to pupils of every age. Families, therefore, will find this Apparatus an invaluable source of instruction for their children, and its introduction into the domestic circle, cannot fail to render a science, which has too often inspired aversion and disgust, the subject of daily amusement and satisfaction.

"This apparatus is very simple, and we cannot but look upon it as one of the improvements of the age."—*Magazine of Useful Knowledge, for May*, 1831. "This apparatus should be used in every family and school where elementary instruction is given."—*New York Evening Journal.* "We have been exceedingly gratified in examining an apparatus, intended for illustrating the general principles of Arithmetic, invented by Mr. Oliver A. Shaw, a practical teacher, of Richmond, Va. It has received the decided approbation of many gentlemen of literature and science, but it needs them not, to commend it to any person, who has opportunity to examine it, and witness its use and application for a few minutes. By this means all the fundamental rules of Arithmetic may be taught with great facility; and, what is far better, the *reason* of the process is explained to every common capacity. The invention is perfectly simple, and every one is surprised that it has not been thought of before."—*Education Reporter, of July 21st*, 1831.

The following testimonials of gentlemen belonging to New York, and others from different parts of the country, who have examined the Visible Numerator, are annexed, together with a letter from Mr. Walter R. Johnson, Principal of the High School in Philadelphia, and Professor of Mathematics in the Franklin Institute of that city:

Philadelphia, May 2, 1831.

Mr. Oliver A. Shaw,

Dear Sir,—Having been favored with a view and explanation of your method of elucidating the elements of arithmetical calculation, by means of sensible objects, I am fully of opinion, that it is superior to any of the devices designed for similar purposes, which have hitherto fallen under my notice. I am persuaded that by the use of the simple means which you apply, the mind of the pupil will be early led to attach the right value to figures according to their *places;* and to comprehend, with perfect clearness, the reason of every step demanded by the nature of the operation which he is to perform by means of figures. By the method of teaching yet extensively in use, the progress of the scholar is chiefly impeded by the mere technical expressions employed about this subject, which, if not left wholly without elucidation, are often explained in terms equally technical, and equally needing explanation. Your *Apparatus*, embracing the combined view of numbers and magnitudes, and their relation to each other, gives an intuitive perception of the truth and propriety of every process required in the solution of a problem. Knowing from long observation, that Arithmetic may be rendered one of the most engaging of pursuits for the youthful mind, I feel assured, that with the aid of such facilities as are presented in this method, no scholar of ordinary capacity need be embarrassed by any of those appalling obstacles, which have too often rendered the whole subject a matter of perplexity and disgust.

I am, dear sir, with high respect, your friend and servant,

WALTER R. JOHNSON.

The following letter is from Mr. W. J. Adams, of Barclay street, New York:

Mr. O. A. Shaw,

Dear Sir,—It is with pleasure that I express my concurrence with Mr. Johnson, in regard to your Apparatus for explaining written Arithmetic. I have never seen anything which appears to me so well designed to impart clear ideas of the relations of different quantities to each other, particularly of different *magnitudes*. As a valuable aid, also, in demonstrating *all* the rules of Decimal Arithmetic, I cordially and confidently recommend "the Visible Numerator" to the notice of every teacher.

I remain, sir, your friend, respectfully,

W. J. Adams.

We fully concur in the opinion expressed by Mr. Adams respecting "*The Visible Numerator*."

John Griscom, Wm. Forrest, S. P. Parker, John Morgan, Joseph C. Hart, S. Johnston, W. P. Lyon, A. P. Stout, Chas. W. Weeks, Leonard Hazeltine, A. Mills, Wm. Sherwood, Joseph Bailey, Noble Heath, Wm. H. Griscom, A. Benedict, S Westcott, Teachers and Literary Professors in New York.

J. Cleveland, James Leach, Charles Atwood, Charles R. Alsop, Wm. Emerson, F. W. Burke, Theodore D. Woolsey, Edward A. Strong, Esquires, and Dr. John R. Knox, New York. Professor A. J. Yates, of the Polytechnic School, Chitteningo, New York. Frederick Emerson, Author of the North American Arithmetic, Boston, Mass. Forrest Shepperd, and A. N. Skinner, Esquires, New Haven, Ct. John Neal, Grenville Mellen, John D. Kinsman, Esquires, Portland, Maine. N. Sargent, Editor of the Magazine of Useful Knowledge, New York. J. L. Van Doren, of the Collegiate Institute, Brooklyn. Levinus Monson Esq., of Delhi, Delaware county, New York. B. B. Blair, Esq., Salem, Washington county, N. Y. Dr. Heman Humphrey, President of Amherst College, Mass.; Professor Sylvester Hovey, of do. B. D. Emerson, Author of the National Spelling Book, Boston, Mass.

Boston, July 19th, 1831.

I entirely concur in the opinion expressed by Mr. Johnson in relation to Mr. Shaw's Visible Numerator.

Solomon P. Miles,
Principal of the Boston High School.

Boston, Sept. 10th, 1831.

Mr. Shaw's *Visible Numerator* appears to me to be a very simple and beautiful contrivance for the purpose of giving to children a clear apprehension of the elements of Mathematics. With a very little explanation, such as any teacher who understands it, or the science of Arithmetic, is perfectly able to give, it cannot fail of impressing upon the young mind, a distinct idea of the relative value of numbers and quantities, whether integral or fractional, and must be much more satisfactory to pupils, than an ordinary *demonstration*, presented by means of numerical characters can be, as to the reason of those processes which are prescribed in the *Rules* usually laid down in arithmetical books. So far as I am capable of judging, no school ought to be without something of the kind,—(and I cannot conceive anything of the kind more simple than this)—unless it be some school where both teacher and pupil are satisfied with learning words without having any ideas associated with them; and rules, without knowing any reason for them.

John Pierpont,
Author of the American First Class Book, &c. &c.

Boston, Oct. 12th, 1831.

I fully agree with the sentiments expressed by Messrs. W. R. Johnson, W. J. Adams, and Solomon P. Miles, in relation to Mr. Shaw's Visible Numerator.

Francis J. Grund,
Professor of Mathematics in the Chauncey Hall School, Boston, and author of the Improved system of Geometry.

In the course of the last Winter and Spring, the Patentee has received numerous testimonials certifying the high utility of the Visible Numerator. The preceding recommendations, however, will be sufficient to call the attention of persons interested in early education, either in the family, or in the school, to the merits of this Apparatus.

The price of the Apparatus is $5 and $8, according to the finish. For sale wholesale and retail, by

RICHARDSON, LORD & HOLBROOK, Boston,

ELAM BLISS, 111, *Broadway, New York,*

And by Booksellers in other parts of the United States.

ADVERTISEMENT.

The Visible Numerator, for which a Patent has subsequently been obtained, was invented in Richmond, Va. during the month of February, 1831, by the subscriber, after having been engaged in the profession of an instructor for upwards of fifteen years, and after having devoted much and long continued attention to the subject of *teaching Arithmetic.*

Among the considerations which prompted the inventor to ascertain, if possible, a philosophical method of imparting to learners an adequate idea of Arithmetic, it occurred to him, that Geometry, or "*the science of extension*," which is an *abstract* science, had, from time immemorial, been taught, and *successfully* taught, by the *visible illustrations* denominated *Diagrams;* that *Arithmetic*, or "*the science of numbers*," which is also an *abstract* science, had ever been taught, but *unsuccessfully* taught, by means of a *descriptive* nature addressed chiefly to the *ear;* the *eye* never having been employed, except in a manner perfectly *arbitrary*, and totally unconnected with the *rationale* of the science; that, while Geometry had been, to almost every pupil, a subject of clearness and satisfaction, Arithmetic, though equally simple in its nature, and less numerous in its principles, had, with few exceptions, been the perpetual source of obscurity and disgust; and that this difference in the success with which these two sciences had been cultivated, had originated in the circumstance, that, while Geometry had been taught *philosophically* by means of *visible objects*, Arithmetic had been pursued precisely as Geometry would have been, had *Simpson* and *Playfair* been stripped of their diagrams, and been suffered to retain nothing but the Theorems of Euclid announced, and his propositions demonstrated with letters alone; and the student thus been doomed to *commit to memory* those Geometrical truths, the nature of which he could not *understand.*

These were some of the considerations, which induced the subscriber to attempt an improvement in the mode of teaching Arithmetic. How far the attempt has succeeded, may best be learned from the fact, that the *Visible Numerator* has, within the few months during which it has been before the public, been adopted by upwards of four hundred of the most distinguished schools in the United States, as well as by many families, who feel interested in the subject of domestic education.

It is, indeed, a matter of surprise, that, while *visible objects* are employed in teaching almost every thing else, Arithmetic, the most practical, and, in our own country certainly, the most extensively important branch of common education, should not be equally favored; that while Geography has its *maps*, Astronomy its *orreries*, Natural History its *pictures*, Geometry its *diagrams*, and even Penmanship and English Grammar have not been without similar facilities, the attempt should still be made, to impart to the youthful mind a knowledge of Arithmetic, by means of the *descriptive*, or, what may with more propriety be termed, the *prescriptive*, *process*.

It will be found, on examination, that the *Visible Numerator* does not supersede any one of the many excellent Treatises on Arithmetic now in general use; and, that it merely exhibits to the *eye*, what they describe to the *ear*.

This apparatus has received the extensive patronage alluded to, aided by no other directions for its use, than those contained in a meagre pamphlet of sixteen pages; and it has recently been a desideratum with the subscriber, to make the instructions concerning its nature and application so ample and plain, that, if occasion should require, a child might use the apparatus, without infringing upon the time of either the parent or the teacher. The former pamphlet of directions, therefore, has been wholly incorporated with the preface of the present work, and speculations of a metaphysical, or an abstruse character, have been entirely excluded from the subsequent "*Description of the Visible Numerator and Instructions for its Use.*" The learner, therefore, will do well to attend to the "*Description*" and "*Instructions*," before he reads the "*Prefatory Remarks*," since he will thus better comprehend the *philosophical principles*, on which this invention is founded. He will also observe, that the delineations of the solids in the *Plates*, are on a scale of *one fourth* the size of the solids themselves. This circumstance, of which due notice will subsequently be given, contributes much to the neatness and the convenient use of the plates; obviates the awkwardness incident to the folding and unfolding of unwieldly sheets of diagrams, and enables the eye to survey, at a single glance, several operations, which would otherwise be liable to confusion, in consequence of not falling, at once, entirely within the scope of the vision.

Great pains have been taken, and much expense has been incurred, in rendering both the book of *Instructions*, and the *Plates*, perspicuous and entertaining; for it is only by presenting the subject of Arithmetic to the pupil divested of the mystery, which has been made ordinarily to envelope it, and by imparting to it, the same degree of interest, which attends the acquisition of many other branches of knowledge, that we can rationally expect the science of numbers to become as feasible in its attainment, as it is useful in its application.

OLIVER A. SHAW.

May 1, 1832.

VISIBLE NUMERATOR.

PREFATORY REMARKS.*

MANY are the improvements which have been made within a few years past in text books on Arithmetic, and it would seem, that after the important benefits bestowed on the cause of education in this department of science, by numerous treatises recently adopted in our schools, little remains to be done, which can be effected by books alone. Colburn, Emerson, and Smith, to say nothing of other authors whose names stand deservedly high in this branch of learning, appear to have advanced as far as mere *description* can go, to impart to pupils a competent and satisfactory knowledge of the principles of Arithmetic. Their works must continue to be used by every intelligent teacher, as the best guides of the youthful intellect through the mazes of numeral calculation. These mazes, however, must continue in a measure *thorny*, as they always have been to a most lamentable degree, until we can render the study of Arithmetic, by some peculiar mode of instruction, which is philosophical in its nature, and simple in its adaptation, as much an object of interest and attraction, as it has been hitherto to most pupils a fruitful source of per-

* The "*Prefatory Remarks*" are chiefly designed to present to Parents and Teachers, a general view of the philosophical principles on which the Invention, termed the "*Visible Numerator*," is founded. *Learners*, therefore, are directed to attend particularly to the "*Description*" and "*Instructions*" previously to reading these "Remarks."

plexity, aversion, and disgust. The invention which has been termed the "VISIBLE NUMERATOR," is founded, as has been previously intimated, on purely philosophical principles. Predicated in its construction and use on the doctrines of intellectual philosophy, as advanced by Mr. Locke, and explained and defended by later metaphysicians, concerning the origin of our abstract ideas, and the mode in which these ideas are introduced into the mind, and eminently simple in its application, this apparatus must prove an essential aid in illustrating those processes of thought so ably *described* and marked out in many of the books on Arithmetic to which we have alluded. It has, indeed, been supposed, by persons who have given much attention to the subject, that the truths of mathematical science, inasmuch as they are eternal and *independent* in their nature, cannot be advantageously taught by means of *sensible* objects: as if the nature of the truths of a science had anything to do with the *mode* in which a knowledge of that science is communicated to the mind. The principles of *geometry* are eternally and independently true; and would have remained forever certain, independently of the existence of the external universe. It would have been an incontrovertible proposition, that "*the three interior angles of a triangle are equal to two right angles*," had the world never been created, nor man inspired with the breath of life—had a triangle of any sort never existed, nor Euclid ever lived. But he who could acquire an adequate knowledge of the geometrical truth contained in the aforesaid proposition, without a sensible illustration, must possess an intellect endowed with a power of acquiring ideas in a way never yet explored, nor entered upon by the mind of man. The nature of a science, then, is one thing; the mode of acquiring a knowledge of that science, another.

The ideas of the pupil must obviously become clear and adequate, in proportion to the simplicity and directness of the means employed to impart those ideas to the mind; and as, in the language of Addison, "*the sight is the best and the noblest of all our senses*," every abstract truth, of an illustration of which the "*faithful eyes*" can take *cognizance*, must be presented to the "*mind's eye*" more speedily and vividly than that truth can be known by any *descriptive process*, or by the exercise of any other *sense*.

In regard to the subject of imparting to the mind a knowledge of the abstract sciences, by means of sensible illustrations, it is incontrovertibly true, that there never has been, and never can be, *any method* of instruction invented, or used, which does not consist essentially in making use of sensible illustrations of the principles of science. It is, in short, metaphysically impossible, for a learner to acquire knowledge from any external source without employing one or more of his *senses* in such acquisition. The very process of description itself, so often used, and by some so tenaciously insisted on, is nothing more than a combination, or a set of *objects* addressed to the *ear* or *sense of hearing*. The only question, after all, is, whether *visible* or *audible* illustrations shall be used, or whether the *ears* or the *eyes* shall be employed in communicating knowledge to the mind. The middle course, however, in this, as in most other cases, is not only the *safest* but the *best;* and the more senses we can enlist in obtaining an acquaintance with any branch of learning, the better.

The idea, which either ought to, or does, enter the mind of the learner, in taking the first step in the science of arithmetic, or in passing from *units* to *tens*, is the precise relation in point of *value*, which the first order of units sustains to the second. This relation is very commonly, though very inadequately, shown by placing to the left of one of any object *ten* objects of the same kind and size, or quantity; as putting by itself a ball on the *abacus* or numeral frame, on one wire, and ten balls on the next, and teaching the pupil to say, *units*, *tens*, &c. By this illustration the mind is improperly led to compare the first order of units with ten times as *many* of itself, whereas, the comparison should be instituted between the unit of the first order and a unit of the second order, or between one order of units of a certain value, and another order of units of a value ten times as *great*. The different *values* of the places in notation are intuitively imparted to the mind, then, by placing before the eye homogeneous objects, each of which is ten times as large as that which precedes it: and since our notion of *value* is primarily suggested by that of *size;* and inasmuch as *size* is more readily and accurately determined by the *sight*, than by any of the other senses, it follows, that a succession of six solids corresponding in *size* to the respective *values* of the six places of the English method of notation, must

present to the mind, at a single view, a clear and an adequate idea of the doctrine of numeration. It is manifest, also, that the French method of notation, where three places belong to the period, can be illustrated in a similar way. In many instances, indeed, the French method of notation is used on account of its greater simplicity, instead of the English. But, by means of the illustrations made with the Visible Numerator, the latter method can be rendered quite as easy as the former.

The Visible Numerator, as will be seen in the sequel, consists of two different sets of solids, one for the tenfold or decimal ratio, adapted to the illustrations of the principles of the five *simple rules*, as they are called in most of the treatises on Arithmetic, viz. Numeration, Addition, Subtraction, Multiplication, and Division, and the corresponding rules of Decimal Fractions, together with the various doctrines of *proportion;* and another set of solids, as will be obvious on inspection, varied in such a manner in point of size, as to accord with the four denominations of sterling money, *farthings*, *pence*, *shillings*, and *pounds;* calculated to exhibit the principle of operating in the *compound* rules and reduction in that species of currency; and, by analogy, to suggest the mode of proceeding with the various other coins, weights, and measures, which are concerned in the same rules. The calculations of federal money are managed with solids of the decimal ratio. Both sets of solids may be used either in combination, or separately, for illustrating the different operations concerned in the reduction, addition, subtraction, multiplication, and division of vulgar fractions.

In families, each pupil should be provided with a slate and pencil, and write down the examples to be illustrated in *figures*, immediately after the illustration is made with the solids on a table.

In using this apparatus, in schools, it will be well for the solids to be exhibited on a table of convenient height, the instructor standing between the table and a *black board*, stationed behind him, for the purpose of receiving *copies* of the abstract operations performed in the mind of the pupil, who sits before the table, and witnesses the illustrations which are made upon it. Here, then, is presented simultaneously the whole business of Arithmetic. The illustration is made with the Numerator on the table, the

abstraction takes place in the mind of the learner, and is immediately copied, either in words or in figures, on the black board, by the learner or by the teacher. *Notation* is illustrated by placing the smallest of the solids, or that which represents units, on the table, and arranging the other solids of the same set, as they increase in a tenfold ratio, towards the left; and here it may be remarked, that the instructor will be under the necessity of putting the solids, as it regards himself, in the position he stands, behind the table, increasing from the left to the right, and this order must be preserved throughout the illustrations, that the *pupils* may view the illustrations in the proper arrangements in the situation *they* occupy in the front of the table. The solids belonging to the set in question, are of six different magnitudes, presenting at once an idea of the manner in which figures occupying the six places of the English notation increase in point of the value which they express, from units to hundreds of thousands inclusively. We thus have the whole *period*, consisting of six *places:* and this period is denominated *units*, *millions*, *billions*, *trillions*, *quadrillions*, &c., according as it is, the first, second, third, fourth, or fifth period from the right. Any pupil may be thus taught with the greatest rapidity to numerate any combination of figures he chooses; and if acquainted (and if not, he readily can be made so) with the derivation from the Latin language of the terms billion, trillion, &c., he may soon give the correct reading, without having occasion to write it down, of any number of figures of the same name, supposed to be written in a horizontal row, from right to left. The term *billion*, he will perceive, is derived from *bis*, which signifies *twice*, and by the association arising from the origin of the word, the mind would be led to assign the name of billion to the second period from the right; whereas, in consequence of the intervention of the term million, between the period of units, and that of *billions*, the idea suggested by the term *billions* falls precisely one period behind that, which it is *arbitrarily* made to signify. A similar remark may be made with regard to trillions, being derived from *tres*, *three*, though actually made to denote the *fourth* period. If, then, the number of one hundred figures were proposed to be read in this *extemporaneous* manner, every figure being supposed to be a 3, and if the whole number of figures be divided by six,

(the number of *places* in a period,) there will be found to be in the one hundred 3s, *sixteen* periods, and four 3s over, for the *seventeenth* period. Now, taking advantage of the principle of derivation previously alluded to, entitling the *seventeenth* period *sex*-decillions, and considering, that every *place* in numeration is either the place of units, or tens, or hundreds, or thousands, or tens of thousands, or hundred of thousands of that *period* to which it belongs, accordingly as it is the first, second, third, fourth, fifth or sixth from the right, the four 3s in the *seventeenth* period must be read three thousand three hundred and thirty-three *sex*-decillions, the sixteenth period, three hundred and thirty-three thousand three hundred and thirty-three quin-decillions, and so on through quatuor-decillions, tre-decillions, duo-decillions, un-decillions, decillions, nonillions, octillions, septillions, sextillions, quintillions, quadrillions, trillions, billions, millions, and units. By an example or two of this nature, beginning in some instances, perhaps, with a smaller number of figures at a time, the pupil becomes so perfectly familiarized with what may be called, with propriety, the *language of figures*, as in no instance to mistake ITS import, nor to be ignorant of THEIR value.

The foundation of arithmetical science being thus laid, in a thorough knowledge of the principles of numeration, the pupil then proceeds to addition: and here, as in all the subsequent doctrines of numbers, the mode of using the *Numerator*, may be best understood by the bare propounding of a concise example under each rule. If it be required, for instance, to perform the addition of two hundred and fifty-three, to sixty-one, the pupil may be asked, how many hundreds in two hundred and fifty-three? He will, of course, answer *two*, then in answer to similar questions concerning the numbers of tens and units respectively, he will reply five tens and three units. The solids representing the two hundred and fifty-three, may be thrown successively in a *promiscuous heap*, without any regard to their order, as denoting units, tens, and hundreds. The pupil being led by a view of them in a previous illustration of Numeration to assign to them their supposed values, will immediately read them, as *two hundred and fifty-three.*

In order to give a clear idea to the pupil, in this stage of the process, of the most expeditious method of copying

the *idea* of two hundred and fifty-three, which has now become abstract in his mind, as well as to show, at the same time, the *necessity* which exists, of pursuing the proper or *conventional* order in writing figures, the figures 2, 3, and 5 may be written at random on the black board, thus, for instance,

3
2,
5

and the pupil requested to read them. He will soon perceive, however, that, written in this manner, they do not present a copy of the idea in his mind, and, therefore, as they are thus written, cannot answer their true purpose; and, indeed, can be read as nothing else than 2, 3, and 5. It will be best, then, to take an *inventory* of the solids on the table, proceeding according to the mode of writing the two hundred, five tens, and three units, or two hundred and fifty-three, which has been *agreed upon*, and authorized by custom, and which has become *conventional* among mankind. The first question in taking this inventory will be, how many units are there in the unit's place of two hundred and fifty-three? The answer being three, the three solids denoting the three units are placed by themselves, and the figure 3 written on the black board. The next inquiry being, how many tens to go into the ten's place in two hundred and fifty-three, and the answer to this inquiry being five, the figure 5 is written at the left hand of the 3, and after a similar question and answer concerning the hundreds, the figure 2 is written at the left hand of the three, making 253, an exact copy, in *figures*, of the idea in the mind of the pupil. It may, perhaps, not be amiss to pursue this digression, which belongs to the subject of Numeration, rather than to that of Addition, a little farther. To exhibit clearly the steps requisite in writing any number in figures, let the solids representing the five tens be removed from the table, leaving only the two hundred, and the three units, or two hundred and three. Now, the idea derived by the pupil from the existing illustration, is two hundred and three; but if the figure 3 is put upon the black board, and the figure 2 immediately at its left, these two figures, thus arranged, will not, according to the principles previously established, amount to a *copy* of the *idea* of two hundred and three, but must be read *twenty-three*. It will thus be seen by the pupil, that the pre-

ceding inquiry must be made with respect to *every place in succession;* thus, how many units for the unit's place; and the figure 3 set down on the black board,—then, *not* how many hundreds for the hundred's place,—but how many tens in the ten's place. The answer being *none,* the character 0 must be put at the left hand of the 3, to signify *none of tens;* the next question will be, how many hundreds? and the answer being two, the figure 2 is set down at the left of the *cypher.* The copy of the idea of two hundred and three is thus completed. The pupil, by this short digression from the main subject under illustration, will have thus acquired, with very little practice, a clear idea of the mode of writing any number whatever.

To proceed now with our illustration of the principles of Addition: the number 253 being thus written down, after placing the representative solids in their proper order on the table, the six *tens* and *one unit,* which constitute the 61, are then placed under the 253, the unit solid being put, of course, directly under, and added to the three unit solids of the 253, and the six solids of *ten* treated in a similar manner. It will be found, that when the three units in the 253 are added to the one unit in the 61, the sum will be four units, and that when the five tens in the 253 are added to the six tens in the 61, the sum will be eleven *tens,* which sum may be more conveniently represented by putting down a solid underneath these tens, which denotes ten tens, or *one hundred,* and another solid of ten, together making eleven tens, than by going to the trouble of counting out eleven individual solids of ten. We then have two hundreds to put down in the hundred's place, which, with the one hundred to *carry* from the last addition, will amount to three hundred, and the whole sum will stand 314; and the operation, from which it arises on the table, may be simultaneously copied on the black board.

To illustrate *Subtraction,* we may take as our first example the same numbers;—and we shall be required to take 61 from 253. The *one* may be taken from the *three* without any inconvenience; but we have, in the next place, to take 6 tens from 5 tens, which apparently cannot be done; and, in order to illustrate to the eye of the pupil, the operation which must occur in the mind in order to accomplish this, one of the hundreds must be removed from the table, and 10 *tens* substituted in its stead;

showing that 1 hundred, 15 tens, and 3 units, are the same thing as 253. The illustration can then proceed with ease, and the 9 tens as a remainder put on the table, with the 1 hundred at its left hand, making 192 for a *remainder*. And all this suggests the *principle* on which the *technicality* of many of the books is founded, in saying 6 from 5 I cannot, but 6 from 15 leaves *nine*, then 1 to *carry* to 0 is 1, and taken from 2 leaves 1, that is as before, 192. Another example which will be given in Subtraction, and which will be sufficient for the purpose of giving a competent idea of the mode of applying the Numerator to the developement of the other principles of this rule, is, to subtract 3 from 200. Here it will be seen, that in order to subtract the 3 units from the 2 hundreds, we must, in the first place, take from the table one of the solids representing hundreds, and substitute ten solids of ten, and secondly, after removing one of these tens, put in its place ten solids of units. All this, when read, will be called 1 hundred, 9 tens, and 10 units. The operation can then be pursued, and the principle illustrated without difficulty.

For Multiplication, one or two examples will suffice to show the manner in which the Numerator is used in this rule; and other examples can of course be given, either in this or other rules of arithmetic, according to the discretion of the teacher, which must be guided by a consideration of the age and advancement of the pupil. Let it be required, then, to multiply 232 by 22. Placing the solids on the table, in a manner corresponding to the usual position of the numbers concerned, as they are written on the black board in figures, thus

$$\begin{array}{r} 232 \\ 22 \\ \hline \end{array}$$

; the operation may proceed by multiplying the 2 units in the multiplicand by the 2 units in the multiplier, which will produce 4 units; then the 3 tens, which will make 6 tens; then the 2 hundreds, which will be four hundred; making 464 for the product of the 232 by the units in the multiplier. The next step will be, to multiply the multiplicand by the 2 tens in the multiplier. Now if we multiply the 2 units in the multiplicand by the 2 tens in the multiplier, the result will be 4 tens; which, in order to place them properly for the purpose of adding the product of

the multiplicand by the 2 tens to the product of the multiplicand by the 2 units, after both multiplications have been performed, it will manifestly be correct to put the solids representing this first product by the 2 *tens* in the multiplier, under the *tens* in the product of the multiplicand by the units: thus showing the reason and meaning of the technicality of the books, which direct the pupil "*to set down the right hand figure of each successive product, one place farther to the left.*"

The next step in this operation is, to multiply the 3 tens in the multiplicand by the 2 tens in the multiplier. Here, the idea of multiplying tens by tens, or solids representing tens by similar solids representing tens, may strike the minds of some teachers as somewhat incongruous, and as being a little like the paradoxical multiplication, occasionally propounded, of two shillings and sixpence by itself. To obviate the appearance of incongruity in the present case, it may be remarked, that, if thought advisable, the *multiplier*, in this example, may be written down in *figures*, as well on the table as on the black board; for, by this time, the pupil will have learned enough of the nature of *figurative* expressions to know the meaning of 22 expressed in figures, as well as the same number represented by the solids. But, inasmuch as it is perfectly correct to multiply 2 by 2 *abstractly*, and we are now concerned in abstract operations, because 2 or any other number of units of the *second order* conveys to the mind an *abstract* idea, as much as the same, or any number of units, of the *first order*, and, inasmuch as the solids are the mere *representatives* of these abstract ideas, there cannot be any thing incongruous, paradoxical, or absurd, in representing the multiplication of units by tens, or even *tens* by *tens*, by means of solids designed to illustrate operations, which are, after all, abstractions of the mind, derived from, or illustrated by, external objects.

After the multiplication of the units in the multiplicand by the tens in the multiplier, the 3 tens and afterwards the 2 hundreds in the multiplicand are to be multiplied by the 2 tens in the multiplier. The two products of 232, first by the 2 units, and then by the 2 tens in the multiplier, are added together, units to units, tens to tens, &c., and the sum 5104 is the product of 232 multiplied by 22. It will probably have been perceived by this time, that the Visible Numerator is no *thought-saving* apparatus; but

that the illustrations made with it will cause the pupil to *think*, and to think with accuracy and pleasure; and convey to his mind clear and adequate ideas of the meaning of the terms, which he uses in Arithmetic. When, in the immediately preceding example, for instance, he multiplies the 2 units of the multiplicand by the 2 tens in the multiplier, he does not say 4, and not know where to set the 4 down, without some technical, and to him *perfectly arbitrary*, direction to put it one remove farther toward the left; but he distinctly recognizes the fact, that the *twice two*, in this multiplication of units by tens, conveys to the mind a very different idea from that suggested by the product of the 2 units by 2 units, when, in the commencement of this operation, he said twice 2 are 4, and set down the 4 as 4 units. He now perceives, that since he is multiplying the units by tens, the product must be *tens*, and that the twice 2 here means 4 tens, which he naturally, and in obedience to the dictates of common sense, puts down under the tens of the first product, and places the subsequent products in their proper order and situation. The doctrine of annexing as many cyphers at the right hand of the product as there are at the right hand of multiplier, or of multiplicand, or of both together, may be explained by multiplying 232 by 220, then 2320 by 22, lastly 320 by 220. It will be perceived in the first case, that the 232 is in fact multiplied by 22 tens; in the second, that 232 tens are multiplied by 22 units; and in the third, that 32 tens are multiplied by 22 tens, and that hence, in each of these cases respectively, as many cyphers must be brought down and annexed to the products by the figures, as are requisite to signify that there are no units in the unit's place, and no tens nor units in each of their places, as the case may be. In all these instances, as in all others, the pupil must inquire *what* he is multiplying, and *by* what he is multiplying, and set the product down accordingly.

Division may be best illustrated by reversing the preceding operations in multiplication. Representing, for instance, 5104 by means of the solids, placed in their proper order on the table; thus, 5 thousand, 1 hundred, and 4 units, constituting a *dividend*, and 2 tens and 2 units (or if preferred, figures may be employed) being put in convenient position to denote the divisor, and the operation pursued according to the

principles of "*Long Division*," it will be found that 232 will be the quotient, which may be represented as it rises by two solids, denoting two hundreds, and three solids, denoting three tens, and two solids, denoting two units, demonstrating to the satisfaction of the pupil, that, if 5 thousand, 1 hundred, and 4 units be divided into 22 equal parts, each of those parts will amount to two hundreds, three tens, and two units.

The *Compound Rules*, including *Reduction*, may be fully apprehended by an illustration in sterling money, of the distinctive principle, which pervades these rules, viz. that in *borrowing, carrying, &c, we should have regard to the number of any preceding denomination, which it takes to make one of the next greater.* As was previously remarked, a part of the Visible Numerator consists of a set of solids illustrative of farthings, pounds, shillings, and pence. The principle just adverted to being ascertained, in relation to this species of currency, every obstacle will be removed as to the other coins, weights and measures, the tables of which are found in most of our Arithmetics.

In Decimal Fractions, it will be convenient to let the solid, which represents tens of thousands in whole numbers, denote the unit; then the solid of one hundred thousand will indicate the ten, and the solid before standing for a thousand will now stand for one tenth, the next less for one hundredth, the next for one thousandth, and the last for one ten-thousandth. It will thus be seen that, by proceeding with the solids in decimal fractions, precisely as has been directed in whole numbers, "*mutatis mutandis*," the subjects of numeration, addition, subtraction, multiplication, and division, of decimals can be illustrated in like manner, since the *diminution* and the *increase* of value is *exactly* the same. The pointing off of decimal places in all these rules, whether of the sum in addition, of the remainder in subtraction, the product in multiplication, and of the quotient in division, must be clearly seen, as soon as these different operations are illustrated, by means of the solids, in correspondence with the *figures* taken in their usual position. Such facts or principles as that ,1 multiplied by ,1 produces ,01, and that ,01 divided by ,1 gives a quotient of ,1, &c., will, of course, be taken into consideration by the person who instructs, and every illustration so made as to suit the age and capacity of the pupil.

The illustration of every principle of *Vulgar Fractions* will be readily accomplished by combining and separating the different solids, as directed by the modes of the Reduction of Fractions, and of their Addition, Subtraction, Multiplication, and Division. One illustration, by way of example, will abundantly suffice to give the teacher an idea of the manner in which the Visible Numerator is applied to this branch of Arithmetic. Let it be required to find the value of $\frac{3}{4}$ of $\frac{1}{12}$, and it will be found to equal $\frac{3}{48}$ or $\frac{1}{16}$. If we take one of the solids which represent shillings for the unit, a solid which denotes a penny will represent $\frac{1}{12}$, and three fourths of this $\frac{1}{12}$ will be three of the solids which represent farthings; and these will amount to $\frac{3}{48}$ of the unit, because one of them taken 48 times, will constitute a solid as large as that which in this instance stands for the unit, or if these three quarters be taken *collectively*, sixteen times, they will amount to the same thing. Other examples might be given, for the sake of illustration, of adding vulgar fractions together, and of subtracting them from, or of multiplying and dividing them by, each other, but it cannot be necessary. These essential operations, as well as the preparatory process of reducing a series of vulgar fractions to a common denominator, &c., will readily suggest themselves in the progress of instruction.

Proportion, and, as one form of this department of mathematical science is called in the books, the *Rule of Three*, may, in every modification of its principles, be illustrated by the Visible Numerator. As an instance, we shall take the solids which denote 10, 100, 1000, and 10000, and place them in their order as the first, second, third, and fourth term of a proportion. It will be seen, that if the solid denoting the fourth term be removed from the table, it will be brought back again by supposing the third multiplied by the second, and divided by the first, as it is directed in our text books on arithmetic. For if the solid representing 1000, be multiplied by, or in other words taken as many times as there are units in the solid representing 100, the product will be represented by the solid of 100000, and if this solid be supposed to be divided by the solid of 10, or more intelligibly perhaps to the pupil, be divided into as many parts as there are units in the solid of 10, one of these parts will equal the solid denoting 10000. The other principles of the doctrine of

proportion may also be demonstrated; as the proposition that the product of the *means* is equal to the product of the *extremes*, proportion by *alternation*, or that if the first term be to the second as the third is to the fourth, the *first will be to the third as the second is to the fourth*, as well as proportion by *inversion*, &c.

Different modifications and applications of the preceding principles of arithmetic, will be explained by the intelligent teacher, as in his discretion they shall be deemed suitable to the circumstances of the pupil.

The following Description of the Visible Numerator, and the Instructions for its use, are so plain, that a child of an early age can understand them, unaided by his parent or instructor.

DESCRIPTION.

The Visible Numerator is contained in a mahogany case about one foot in length, seven inches broad, and six inches thick. This case, on removing the slide, will be found to enclose *ten solids*, each of which is five inches long, two inches wide, and one inch thick;—*twenty solids*, each of which is five inches long, and one inch square at the end; and *a box* five inches long, four inches wide, and five inches deep, containing other solids of less magnitude, together with a small round box about two inches in diameter, and an inch deep, which contains some solids of the smallest size belonging to the Apparatus. It will be necessary to attend particularly to the following description of the different sizes of these solids. The smallest solid of all will be found in the small round box. This smallest solid is a *cube* measuring one tenth of an inch, on each side, and is called the solid of a *unit*. There are thirty of these solids of a unit in the small round box. In the same box, you will find thirty other solids, each ten times as large as the solid of a unit. One of these is called a solid of *ten*. Among the solids in the larger box, will be found twenty solids, each of which is an inch square and one-tenth of an inch thick, and of course ten times as large as a solid of ten. Each

of these twenty solids is called a solid of a *hundred*. In the same box you will also find fifteen solids, measuring one inch every way, or one cubic inch in dimension. Each of these fifteen solids is ten times as large as a solid of one hundred, and is called a solid of a thousand. If you put ten of these solids of a *thousand* together, five in one line and five in another, they will be as large as one of the largest solids, which you first saw on opening the case, and each of which measures five inches long, two inches wide, and one inch thick. Each of these ten largest solids is called a solid of *ten thousand*. The box, which is five inches long, five inches deep, and four inches wide,* may be called the solid of *one hundred thousand*. (See Plate I. Fig. 1.*) Beside the above mentioned solids, there will be found contained in the box called the solid of one hundred thousand, among the solids of a hundred and solids of a thousand, eight solids of half an inch long, half an inch broad, and a little less than half an inch thick. Each of these eight solids is called a solid of a *farthing*. You will also find in the same box, fourteen solids, each of one inch square, and of the same thickness with the solid of a farthing. One of these fourteen solids is called the solid of a *penny*, and is just four times as large as the solid of a farthing. *Twelve* of these penny solids put together, face to face, will just equal, in size, one of the *twenty* solids, which are found in the case containing the whole Apparatus, and each of which are five inches long, and one inch square at the end. Each of these twenty solids is called a solid of a *shilling*. If these twenty solids be put together, by laying them in such a manner as that the pile shall measure one of these solids, or five inches in length, five of them, or five inches wide, and four of them, or four inches in thickness, the whole collection will be found to be of the same size as the box, which has been called the solid of one hundred thousand.

* It will be borne in mind by the learner, that the delineations of the solids in the Plates are, for the sake of convenience, represented on a scale of *one-fourth* the dimensions of the solids themselves. Thus the delineation of the solid of one hundred thousand, as it appears to the eye in the Plates, is two and a half inches long, two and a half inches deep, and represented, in perspective, as two inches wide; which dimensions make the delineation of the solid just one-fourth as great as those of the solid of one thousand itself. The same will be found true of the solids of the ten thousand, the thousand, the hundred, the ten, and the unit.

This box may be also called the solid of a *pound sterling*. (See Plate I. Fig. 2.*)

The Visible Numerator you thus find to comprise two sets of blocks or solids. The set first described, embraces the solids of a unit, solids of ten, solids of a hundred, solids of a thousand, solids of ten thousand, and the solid of one hundred thousand. This set is intended to explain the *simple* or *decimal* rules of Arithmetic; viz: *Simple Addition*, *Simple Subtraction*, *Simple Multiplication*, *Simple Division*, *Decimal Fractions*, and all the rules, to which *simple* or *decimal Numeration* applies. The second set consists of the solids of a farthing, solids of a penny, solids of a shilling, and the solid of a pound. This second set is designed to illustrate the *compound rules*. The manner, in which both these sets of solids are used, to convey to your mind a clear understanding of the principles of Arithmetic, will be learned by attending to the following

INSTRUCTIONS.

Every child as soon as he can read the figures in Arithmetic, knows, that the figure 1 means *one*; as *one* apple, *one* orange, *one* marble, or *one any thing*; that the figure 2 means *two*; that the figure 3 means *three*; that the figure 4 means *four*; that the figure 5 means *five*; that the figure 6 means *six*; that the figure 7 means *seven*; that the figure 8 means *eight*; that the figure 9 means *nine*; and that the figure 0 or *Zero*, as it is sometimes called, means *nothing* or *none*. The meaning of each of these figures, then, is perfectly plain. Write on your slate the figure 3. You know, and so does every child, who can read, that it means *three* marbles, *three* chairs, *three* houses, or *three any thing else*. So you may write any other figure; and you will perfectly understand what it means. All this, you will say, is plain

* The pupil will find the solids denoting the denominations for Sterling Money also represented in the Plates, on a scale of *one-fourth* the dimensions of these solids themselves. The dimensions of the solid of a hundred thousand, which denotes, likewise, a pound sterling, have been mentioned in the last note. The delineation of a solid of a shilling will be found to be two inches and a half long, half an inch wide; and the perspective or shade is designed to represent it as half an inch thick. This dimension is *one-fourth* of that of the solid itself. The same is true of the solids of a penny and a farthing.

enough; so is the whole of Arithmetic, as soon as it is understood. Now, in order to understand Arithmetic, you must first know where the *difficulty* in Arithmetic *begins.* Now you will say, there is difficulty and obscurity *somewhere* in Arithmetic, though you cannot tell the exact place where this difficulty or obscurity commences. Now I shall point out to you the very spot where you begin to be puzzled in studying Arithmetic; and you will see, when I do point out to you this spot, that the difficulty and obscurity commences *there.*

Supposing, then, that after writing the figure 3 on your slate, or on the black board, you put *five* other 3s at the side of it; making the six 3s stand thus, 333333. These six 3s, thus written, mean *three hundred and thirty-three thousand three hundred and thirty-three.* Now you see, that it is much more difficult to read these six 3s altogether in this manner, than it is to read one 3 by itself. If you write one 3 on the left hand of the other, thus, 33, you will perceive, that it is a little more difficult to read these two 3s as thirty-three, than it is to read one 3; that it is more difficult still to read three 3s, 333 *three hundred and thirty-three,* and so on, for four 3s, 3333; and five 3s, 33333; the obscurity increasing the more 3s you write in the *horizontal position,* or in a *straight line* from right to left, until you come to the six 3s, which are more difficult to read than either 33 *thirty-three,* 333 *three hundred and thirty-three,* 3333 *three thousand three hundred and thirty-three;* or 33333 *thirty-three thousand three hundred and thirty-three.* So you see, that *the obscurity in Arithmetic begins where you* BEGIN *to write the figures in a horizontal row.* The figure 3 written by itself is understood as soon as it is seen; but when several 3s are written, one after another in a horizontal position, (thus, for instance, 3333,) you have to reflect a little before you can read them. Now we shall make it as easy for you to read any number of figures written one after another, in a horizontal direction, as it is to read one figure written by itself. Write on your slate three 3s; thus, 333. Let the 3 on the right hand be called the *first* 3; let the next 3 on the left of the first 3 be called the *second* 3, and let the other 3, or last 3 on the left, be called the *third* 3. Now, you will be surprised, perhaps, if I tell you that the first 3, the second 3, and the third 3, mean, each, precisely the *same*

thing. You will be still more surprised when I tell you, that the first 3, the second 3, and the third 3, mean entirely *different things*. By a little attention, however, you will see, that both these assertions are strictly true, that is, that each of the three 3s has the *same* meaning and also a *different* meaning. Now let us examine this meaning a little. The first 3 means three *units*, the second 3 means three *tens*, and the third 3 means three *hundreds*. Now three hundreds are no more *three* than three tens, and three tens are no more *three* than three units. The fact is, that the three units are *three things*, and neither the three tens, nor the three hundreds are any more than *three things*. The hundreds are three in *number*, the tens are three in *number*, and the units are three in *number*. To make this plain, take three marbles, three carriages, and three houses. Now the three houses are no more than *three things*, neither are the three carriages more than *three things*, and the three marbles are just *three things*. There is the *same number* of marbles, carriages, and houses, viz. *three* marbles, *three* carriages, and *three* houses. So in the 333, there are *three* units, *three* tens, and *three* hundreds. So that the *number* of the units, tens and hundreds is precisely the same, that is, each of our three 3s, as it respects their *number*, means the same thing. Now having seen, that they mean the *same* thing, let us see how it is, that they mean *differently*. If we take again our three houses, three carriages, and three marbles: the three carriages would be worth more, or be more *valuable* than the three marbles, and the three houses would be more *valuable* than the three carriages; and the three tens are ten times as *valuable* as the three units, and the three hundreds are ten times as *valuable* as the three tens. The *value* of the three units, of the three tens, and the three hundreds, then, are all *different*. We thus see, that, as it respects VALUE, our three 3s mean *differently;* while as it respects NUMBER their meaning is the *same.** If, then, you would clearly understand the three 3s when you read them, as three hundred and thirty-

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* This idea of *number* and *value*, the philosophy of our Arabic mode of Notation, compels us to impart clearly to the pupil, if we would successfully use this mode in teaching the science of Arithmetic. Whoever attends to the operations of his own mind will clearly perceive, that at the moment he compares the three units with the three tens, he must, *eo instante*, consider the *value* of each *three*, as denoting three *units* or three *tens*

three, you must consider each 3 as denoting the same *number*, but a different *value* according to the *place* in which it stands. And, if you would know how to write three hundred and thirty-three in *figures* instead of *words*, you must know how to place the 3s so as to mean three hundred and thirty-three. This *reading* and *writing* of figures correctly, is what is called

NUMERATION.

In order to understand *Numeration*, you must write six *ones* in a *horizontal row*, thus **111111**; we generally begin with the one at the right and say *units*, calling the second one from the right *tens;* the third *hundreds;* the fourth *thousands;* the fifth *tens of thousands*, and the sixth *hundreds of thousands.* To commit this to memory is what is commonly called *learning the Numeration Table;* now let us see what this *units*, *tens*, *hundreds*, *thousands*, *tens of thousands*, *hundreds of thousands* MEANS. Each one of these 1s means *one* and no more than *one*. The first 1 at the right means *one unit;* the second 1 from the right means *one ten;* the third 1 from the right means *one hundred;* the fourth 1 from the right means *one* thousand; the fifth 1 from the right means *one* ten thousand; the sixth 1, or the 1 on the left means *one* hundred thousand. Let us call the 1 on the right, the *first* 1; the next 1 to the left, the *second* 1; the next towards the left, the *third* 1; the next 1 to the left, the *fourth* 1; the next towards the left, the *fifth* 1; and the 1 on the left, the *sixth* 1.* What I wish you clearly to understand is this; viz. that the second 1 means ten times as much as the first 1; that the third 1 means ten times as much as the second 1; that the fourth 1 means ten times as much as the third 1; that the fifth 1 means ten times as much as the fourth 1; and that the sixth 1 means ten times as much as the fifth 1; that is, that each 1 which follows towards the left, means ten times as much as the 1 which goes before towards the right. To obtain a clear idea of all this, write the six 1s on a paper, slate, or black board, as you see them written in Plate II. Figure 1. Then place on a table, or desk,

* These 1s are sometimes called, beginning at the right, the first *order* of units, the second *order* of units, the third *order* of units, the fourth *order* of units, the fifth *order* of units, and the sixth *order* of units.

before you, one of the solids of a unit, and it will represent the first 1, or *one unit;* a short distance to the left of this solid of a unit, place a solid of ten, and it will represent the second 1, or one ten;* to the left of this, place a solid of a hundred, and it will represent the third 1, or one hundred; to the left of this, place a solid of a thousand, and it will represent the fourth 1, or one thousand; to the left of this, place a solid of ten thousand, and it will represent the fifth 1, or one ten thousand; to the left of this, place the solid of one hundred thousand, and it will represent the sixth 1, or one hundred thousand. (You will see figures of these solids drawn under the 1s in Plate II. Fig. 1.) We thus have the meaning of *units, tens, hundreds, thousands, tens of thousands, hundreds of thousands,* represented by the six solids, viz. of a unit, of ten, of a hundred, of a thousand, of a ten thousand, and of a hundred thousand. Now ten *units* make *one ten,* ten *tens* make *one hundred,* ten *hundreds* make *one thousand,* ten *thousands* make *one ten thousand,* ten *ten thousands* make *one hundred thousand:* and if you put together ten solids of a unit, they will together be equal to one solid of ten; and, in the same way, you will find ten solids of ten to be equal to one solid of a hundred; ten solids of a hundred equal to one solid of a thousand; ten solids of a thousand equal to one solid of ten thousand; and ten solids of ten thousand equal to the solid of one hundred thousand. The solid of a hundred thousand, then, is a hundred thousand times as large as the solid of a unit; and if we call the solid of a unit, *one unit,* the solid of one hundred thousand will be a hundred thousand *units;* if we call the solid of a unit *one apple,* the solid of a hundred thousand will be a hundred thousand *apples;* if we call the solid of a unit one *million,* the solid of a hundred thousand will be a hundred thousand *millions;* if we call the solid of a unit one *billion,* the solid of a hundred thousand will be a hundred thousand *billions;* if we call the solid of a unit one *trillion,* the solid of a hundred thousand will be a hundred thousand *trillions;* if we call the solid of a unit one *quadrillion,* the solid of a hundred thousand will be a hundred thousand *quadrillions,* and so on

* The solid of ten, which you put on the table or desk IS ten times as much as the solid of a unit;—the second 1 MEANS ten times as much as the first 1. In this way proceeding, the learner cannot fail to acquire an *adequate* idea of the *meaning* of the "*Numeration Table.*"

through quintillions, sextillions, septillions, octillions, nonillions, decillions, and as far as we please. Whatever we call the smallest of the six solids, the largest solid will represent a hundred thousand times as many; so you see that *every thing* is either hundreds of thousands, or tens of thousands, or thousands, or hundreds, or tens, or units, of whatever you call the solid of a *unit;* whether it be a *unit*, or a *million*, or a *billion*, or a *trillion*, or a *quadrillion*, or a *quintillion*, or *any thing else.* If you write six 1s on your slate horizontally, thus 111111, and begin to numerate at the right hand, you will have first a *unit*, secondly a *ten*, thirdly a *hundred*, fourthly a *thousand*, fifthly a *ten thousand*, and sixthly a *hundred thousand.* All the six 1s make what is called a PERIOD in Numeration, and fill six PLACES. Now write twelve 1s in the same manner you did the six 1s, thus 111111111111. Begin at the right and numerate unit, ten, hundred, thousand, ten thousand, hundred thousand. You have now numerated *six* PLACES, or *one* PERIOD;—you may then put a comma after the sixth place and the twelve 1s will appear thus, 111111,111111; then we have numerated and separated the *period* of *units;* the first 1 after the comma is a unit again; but it is a unit of a *million*; the next 1 is ten, but ten of a *million;* the next 1 is a hundred, but a hundred of a *million;* the third 1 is a thousand, but a thousand of a *million;* the next 1 is ten thousand, but a ten thousand of a *million;* the next 1 is a hundred thousand, but a hundred thousand of a *million.* You thus complete the *second period*, or *million's period.* If you should still write 1s to the left, after completing the million's period, the first 1 would be a unit of *billions*, and so on as in the two periods already written. After completing the period for billions, you would begin again with units of *trillions*, &c. You see by this time, that, in *Numerating*, the first *place* of every *period* is the *unit's* place; the second, the *ten's* place; the third, the *hundred's* place; the fourth, the *thousand's* place; the fifth, the *tens of thousand's* place; and the sixth, the *hundreds of thousand's* place.—To make the mind perfectly familiar with the subject of Numeration, we will propose some examples both in the *writing* and *reading* of numbers. And first, we will fix upon the proper method of writing numbers. Let it be proposed to write, in *figures*, the number, *two hundred and fifty-three.* In two

hundred and fifty-three, there are two *hundreds*, five *tens*, and three *units*. Throw, then, upon the table two solids of a hundred, five solids of ten, and three solids of a unit, in a promiscuous heap, without any regard to their *order*, as in Plate II. Figure 2. Now the idea which we obtain from the inspection of these solids, is *two hundred and fifty-three*, or two *hundreds*, five *tens*, and three *units*. Now we wish to make a *copy* of this idea of two hundred and fifty-three, in *figures*. We might, indeed, write it in *words*. But, if we did this, the phrase *two hundred and fifty-three* would consist of upwards of *twenty* different characters, or *letters*;—the word *two* has three letters, viz. T, W and O, and some of the other words in the expression, many more. Now the utility of writing figures, consists in the fact, that, when we use them to express numbers, we have fewer *characters* to make, than if we use *letters* and *words* for that purpose. In the present example, for instance, we can express by the three figures, 2, 3 and 5, as much as we can by writing out in *words*, two hundred and fifty three. Write these three figures, then, on your slate, 2, 5 and 3, *promiscuously*, and without regard to their *order*, as you see in Plate II. Figure 3., *just as you have thrown the solids on the table.* You perceive, that the figures written in this promiscuous manner, form no *copy* of the idea of *two hundred and fifty three.** We will, then, take a *list* or an *inventory* of the solids on the table: and we will begin with the smallest solids. *First*, then, there are 3 solids of a unit, that is, 3 units. Set down the 3 on the slate and place the 3 solids of a unit by themselves on the table as you see in Plate II. Figure 4. *Secondly*, take the solids which are next in magnitude, that is, the 5 solids of *ten*, and put them at the left hand of the 3 solids of a *unit*, and write the figure 5 on the slate at the left of the 3, as in Plate II. Figure 5. *Lastly*,

* It will thus be seen by the pupil, that *figures* are perfectly *arbitrary* in their signification. For he will obtain the idea of two hundred and fifty-three from the view of the solids thrown on the table, without any regard to their order; but he cannot obtain the same idea from the figures 2, 3, and 5, written promiscuously on the slate. The reason of this is, that he, according to the principles which govern his intellectual operations, infers the *value* of the two hundreds, and five tens, and three units, respectively, from the different magnitude of the solids, however confused their position may be; but he has not even a *conventional* guide, to direct him to such an inference, from the *figures* in a similarly disordered condition.

place the 2 solids of a *hundred* at the left hand of the 5 solids of *ten*, and write the figure 2 on the left hand of the 5 accordingly, as you see in Plate II. Figure 6. Thus we have 2 *hundreds*, 5 *tens*, and 3 *units* represented by the *solids*, and the number *two hundred and fifty-three* set down in *figures*. The figures 2, 3, and 5, written thus, 253, are a *true copy* of the *idea* of two hundred and fifty-three. Now, you may place, at the right, the 2 solids of a hundred, (that is, begin with the *largest* solids first, instead of the smallest, as we did before,) then the 5 solids of ten at the left of the 2 solids of a hundred, and, lastly, the 3 solids of a unit at the left of all. You may, likewise, arrange the figures 2, 3, and 5, on your slate, in the same order in which the solids are placed on the table, as you see in Plate II. Figure 7. The figures 2, 3, and 5 placed as they are in Figure 7 of Plate II. answer equally as well to express the idea of two hundred and fifty-three, as the 2, 3, and 5, written as they are in Figure 6 of the same Plate; because the expression "3 *units, 5 tens, and* 2 *hundreds*" conveys to the mind the same idea, which the expression "2 *hundreds*, 5 *tens, and* 3 *units*" does. The *order*, in which the hundreds, tens, and units are mentioned, is not material. The only reason *why* we write the units at the right hand, then the tens at the left of the units, and then the hundreds at the left of the tens, is, because it is *customary;* that is, *mankind have agreed to do so.* In a similar manner, we say, "*two dollars and fifty cents*," and not "*fifty cents and two dollars*." But it is easy to see, that it would be just as proper to say "*fifty cents and two dollars*," as it is to say "*two dollars and fifty cents*," if *custom* had authorized it. You understand, then, that the *order*, in which the figures are written, depends entirely upon *custom*, and, that if the common consent of mankind had sanctioned such a mode of writing numbers, it would be just as correct, to begin to write numbers, by writing the figure of the greatest value first, and then proceeding with those of the less value towards the left, as in Figure 7, Plate II.; or, as in Plate II. Figure 8, by writing them one under the other in a *vertical direction;* or, as in Plate II. Figure 9, by writing them *slantwise* or *obliquely*, or in any other direction, even in the *confused* manner, in which the figure 2, 3, and 5 are written in Plate II. Figure 3.; *as it*

now is, in consequence of the agreement of mankind, to write numbers in the manner illustrated in Plate II. Figure 6, that is, by writing units first at the right hand, then towards the left tens, hundreds, thousands, and so on.

You may now arrange the solids as they are represented in Plate II. Figure 10, and then take away the 5 solids of ten, and let the other solids remain as in Plate II. Figure 11. We have remaining, as you see, 3 solids of a unit, and 2 solids of a hundred, representing 3 units and 2 hundreds. Supposing that we proceed to make a copy of this 3 units and 2 hundreds, by writing the units first, then the hundreds, as in Plate III. Figure 1, we should have, *not* two hundred and three, but *twenty-three;* and yet we began our copy of the number, by writing the units at the right and proceeding with the figures of greater value towards the left, as we did in writing the *two hundred and fifty-three* in our last example. In writing the two hundred and fifty-three, we ask ourselves, In two hundred and fifty three, how many units?* the answer is *three;* we then write the figure 3; we then ask ourselves, In two hundred and fifty-three, how many tens? the answer being *five,* we write the figure 5 at the left hand of the 3; we inquire, lastly, In two hundred and fifty-three, how many hundreds? The answer being *two,* we write the figure 2 at the left hand of the 5; and we have our two hundred and fifty-three expressed in figures, thus, 253.—Now in writing the two hundreds and three units, or two hundred and three, as it is customary to express it, we inquire, In two hundred and three, how many units? the answer being *three,* we write down the figure 3. And then ask, *not* how many hundreds in two hundred and three? but *In two hundred and three, how many tens?* the answer is *none;* we then write *none* or a zero to the left hand of one 3 as in Plate III. Figure 2, to signify no tens; and we have copied *no* tens and 3 units. We then ask ourselves, lastly, In two hundred and three, how many hundreds? the answer being *two,* we put a figure 2 at the left hand of the cypher as in Plate III. Figure 3. We thus have, *two hundred and three* correctly expressed in figures. You see, then, *that when*

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* The pupil will bear in mind, that in this and similar questions, we do not mean how many units in the *whole number* to be written; but how many units in the unit's *place;* how many tens in the ten's *place,* &c.

we wish to write a number, it is necessary to inquire what there is to be written in EVERY PLACE, *as we proceed from right to left*; *and to put a* FIGURE *or a* CYPHER, *accordingly as there is* ANY THING *or* NOTHING *to be written in* EACH PLACE. We will now propose several other examples of numbers, to be written in *figures*. Let it be required to write in figures, *four thousand and fifty*. The first inquiry is, In four thousand and fifty, how many units? The answer is, *none*. We therefore write a cypher to signify no units. The next enquiry is, In four thousand and fifty, how many tens? The answer is *five*, and we write a figure 5 at the left hand of the cypher, which is in the unit's place. We next inquire, In four thousand and fifty, how man hundreds? The answer is *none*, and we put a cypher at the left hand of the 5, which is in the ten's place. We then inquire, In four thousand and fifty, how many thousand? The answer is *four*, and we write a figure 4 at the left hand of the cypher, which is in the hundred's place. We now have, as you see, in Plate III. Figure 4, the four thousand and fifty, expressed in figures, thus, 4050.

Let our next number to be written in figures, be *one hundred and six thousand two hundred and four*. First, then, In one hundred and six thousand two hundred and four, how many units? The answer is *four*. Write the figure 4 in the unit's place. Next, In one hundred and six thousand two hundred and four, how many tens? The answer is *none*; put a cypher, therefore, in the ten's place. Again, In one hundred and six thousand two hundred and four, how many hundreds? The answer is *two*; put a 2, therefore, in the hundred's place. Again, In one hundred and six thousand two hundred and four, how many thousands? The answer is *six*; then place a 6 in the thousand's place. Again, In one hundred and six thousand two hundred and four, how many ten thousands? The answer is *none*. Put a cypher, therefore, in the tens of thousand's place. Lastly, In one hundred and six thousand two hundred and four, how many hundreds of thousands? The answer is *one*. Put 1 in the hundreds of thousand's place. We then have, as in Plate III. Figure 5, the number one hundred and six thousand two hundred and four, expressed in figures; thus, 106204.

You may now represent the last two numbers, which we have

written, by the solids. See Plate III. Figures 6 and 7. In Figure 6 of Plate III. you see the number four thousand and fifty written in the figures 4050. Below the figures are five solids of ten, and at their left four solids of a thousand, *representing* four thousand and fifty. In Figure 7 of Plate III. you see the number one hundred and six thousand two hundred and four, written also in the figures 106204. Below these figures you see delineated one solid of a hundred thousand, six solids of a thousand, two solids of a hundred, and four solids of a unit; placed in an order similar to that of the figures. These solids *represent*, as you see, one hundred and six thousand two hundred and four.

We will now give one more number to be written in figures. Let then this number be *fifty-five billions and eleven.* There is in this number, one unit, one ten, *no* hundreds, *no* thousands, *no* tens of thousands, *no* hundreds of thousands, for the *unit's period;* which is thus filled up, 000011; and neither units, tens, hundreds, thousands, tens of thousands, nor hundreds of thousand, in the *million's* period; the places of which must, of course, be filled with zeros or cyphers, thus 000000; and there are five units and five tens in the *billion's* period, which will be 55. Now we must put the *unit's period* first, then at the left of the unit's period, place the *million's period,* and lastly the *billion's period* at the left of all, thus 55,000000,000011; all which may be read, five tens of billions, five units of billions, *no* hundreds of thousands of millions, *no* tens of thousands of millions, *no* thousands of millions, *no* hundreds of millions, *no* tens of millions, *no* units of millions, no hundreds of thousands of units, *no* tens of thousands of units, *no* thousands of units, *no* hundreds of units, one ten of units, and one unit of units. All this, in reading, is usually shortened and read, *fifty-five billions and eleven.* You will, now, I think, have no difficulty in writing any other number you please.*

We will now give a few numbers expressed in figures to be read in words. 1st. Read in words the figures 507. Now we know, that the first or right hand *place* in every *period* is the unit's place;

* The pupil can represent by the solids, and express other numbers in figures, at pleasure. He will find no difficulty in doing this, after the minute instructions which have been given on this subject.

that the second place or next to the left hand is the ten's place; that the third place is the hundred's place; the fourth place is the thousand's place; that the fifth place is the tens of thousand's place, and that the sixth place is the hundreds of thousand's place; and we also know, that, when there is no more than one period to be read, that period must be the unit's period. In this number, 507, the 5 being in the third place of the period means five *hundreds*; the 0 being the second place means *no tens*, and the 7 being in the first place means 7 *units*, and there being no more than *one period* in the number, the figures 507 mean five hundred of *units*, *no* tens of *units*, and seven units of *units*, or five hundred and seven *units*, which we call, in ordinary language, five hundred and seven; for when, in reading figures, we do not mention the period to which they belong, we take it for granted, that the unit's period is meant. Thus if we say two hundred and forty-six, we understand that two hundred and forty-six *units* is intended, and not two hundred and forty-six millions, nor two hundred and forty-six of any other period than unit's period. Let us now propose to be read, another number expressed in figures. Let this number be 120490. You may illustrate this number with the solids. See Plate IV. Figure 1. You may write the 120490 on your slate, and then put on your table, as you see described in the plate, one solid of a hundred thousand, two solids of ten thousand, four solids of a hundred, and nine solids of ten. You will, then, have one hundred thousand, *two ten* thousands, or *twenty* thousand, four hundreds, and nine tens: That is, as read in the customary language, *one hundred and twenty thousand four hundred and ninety.* Let us read one more number expressed in figures, viz. 107854960392547984. When you have a large number like this to be read, it will be well to divide it into periods. To do this, we must count off in the first place six places from the right, and put a *comma* or some other mark before the sixth place, and these six places will constitute the *unit's period.* In the present example the unit's period will consist of the figures 547984; next to these six figures, count off the million's period and it will consist of the figures 960392, and put a comma before the 9; then count off the billion's period which will consist of the figures 107854; the whole number, thus separated into periods, will stand thus, 107854,960392,547984.

There will now be no difficulty in reading it; for the figures in the *third* period will be, one hundred and seven thousand eight hundred and fifty-four BILLIONS; the *second* period will be, nine hundred and sixty thousand three hundred and ninety-two MILLIONS; and the first period will be five hundred and forty-seven thousand nine hundred and eighty-four UNITS; and the whole will be read, one hundred and seven thousand eight hundred and fifty-four *billions*, nine hundred and sixty thousand three hundred and ninety-two *millions*, five hundred and forty-seven thousand nine hundred and eighty-four.

You may read other examples, and express them by the solids at pleasure. A very little practice in this way will render *Numeration* both familiar and entertaining.

After having obtained an adequate knowledge of the mode of writing and reading numbers, as expressed by figures, the next step is to acquire the art of *putting numbers together*. Numbers are put together in *two different ways*. The first mode of putting numbers together is called *Addition*, and the second *Multiplication*. We shall first give you an idea of

ADDITION.

If you were required to add together 3 units and 4 units, it is plain that the sum would be 7 units, or 7. But supposing you were required to add 7 units and 5 units together, you might write 7 and then 5 immediately under it, and say 5 and 7 make 12. To understand clearly how this is done, see Plate IV. Figure 4. After placing the 7 on the slate, you may write the 5 under it, and draw a line under them both. Now put on the table 7 solids of a unit, and 5 solids of a unit under the 7 solids of a unit; and these solids will represent the 7 units and the 5 units. Then counting the 7 solids of a unit with the 5 solids of a unit, the whole will amount to 12 solids of a unit. To represent these 12 solids of units, put on your table 1 solid of ten for ten of the units, and 2 solids of a unit. You thus have represented by the solids 1 ten and 2 units which is the same thing as 12 units, or the sum of 7 units and 5 units; and it is more convenient to represent this sum, 12 units, by one solid of ten and two solids of a unit, than

it would be to count off 12 solids of a unit for that purpose. You may now add 56 units or *fifty-six*, to 3 units or *three*. You will see this operation illustrated in Plate V. Figure 1, in which the numbers 56 and 3 are written, and then represent the 56 by 5 solids of ten and 6 solids of a unit, and the 3 by 3 solids of a unit, placed, for the sake of convenience, *directly under* the 6 solids of a unit. If they are added together, they amount to 59, or as represented by the solids, to 5 solids of ten and 9 solids of a unit. You may now add together 253 and 61. The way in which Addition is commonly directed to be done in the books on Arithmetic is, *to put the units under units, tens under tens, and draw a line underneath them, thus*

253
61
314

then beginning with the figures at the right, we are told to say, 3 and 1 make 4, which we set down under the figures we add; then 5 and 6 are 11, and set down one of the 1s and say 2 and "*one to carry*" make 3; and the sum of the 253 and 61 is thus found to be 314. Now we wish to know the reason *why* we "carry one," and to make the whole process perfectly plain and rational, we will take 3 solids of a unit, 5 solids of a ten, and 2 solids of a hundred for the 253, and arrange them according to the order of the figures. See Plate IV. Figure 2. We will also take 6 solids of ten and 1 solid of a unit for the 61, and arrange them in a similar manner. Now the object of the operation is, to add the 2 hundreds, 5 tens and 3 units to the 6 tens and 1 unit, and find what they all amount to when added together. We will add the units in the 253 to the units in the 61; the tens in the 253 to the tens in the 61; and join to them the hundreds in the 253, and we shall thus ascertain the *amount* or *sum* of the 253 and the 61. If, then, we add the 3 units in the 253 to the 1 unit in the 61, there will be 4 units, and you may place 4 solids of a unit on the table as you see in Plate IV. Figure 2. Then we find 5 tens in the 253, and 6 tens in the 61, which make 11 tens in both these numbers. We might now place 11 solids of ten on the table for the sum of these 5 tens and 6 tens. But it will be better to put one solid of a hundred, which will be equal to ten solids of ten; and one solid of ten which will make 11 tens; for it is more convenient to

express the sum of our 5 tens and 6 tens by one solid of a hundred, or ten tens, and one solid of ten, than it would be to count off 11 solids of ten for that purpose; for we thus express the same by two solids, that we should by 11 solids, as you see in the Figure 2, Plate IV. We have now added the units in the 253 to the unit in the 61; and the tens in the 253 to the tens in the 61; making 11 tens and 4 units, or, which is the same thing, one hundred and one ten. Now if you place on the table two solids of a hundred for the two hundred in the 253, with the one hundred and one ten and four units, as you see represented in Figure 2 of Plate IV.; you will have, for the whole sum of 253 and 61, two solids of a hundred, and one solid of a hundred, and one solid of ten, and four solids of a unit. Now what is meant by "*carrying*" in the *rules* which you find in the "*Arithmetics,*" is nothing more than taking the solid of one hundred, which you have put by the side of the solid of one ten in order to express the sum of the 5 tens in the 253 and the 6 tens in the 61, and putting it *among*, or *together with*, the two solids of a hundred, which you have placed on the table to signify the two hundreds in the 253. This you see is done in Plate IV. Figure 3. You may then say two hundred and "*one to carry*" is three hundred, taking up this solid of a hundred and "carrying" it to, or putting it with, the two solids of a hundred. So you see, that the 3 solids of a hundred, one solid of ten, and the four solids of a unit, in Figure 3 of Plate IV., are the same thing as the two solids of a hundred, and one solid of a hundred, and one solid of ten, and four solids of a unit, in Plate IV. Figure 2. We have thus *carried* "*one for every ten,*" because, the *one solid of a hundred*, which, with the solid of ten, in Plate IV. Figure 2, represents the sum of the 5 tens in the 253 and the 6 tens in the 61, and which we carry to the two solids of a hundred, is equal to *ten solids of ten*, which we might have counted off, had it been equally convenient, with the one ten to make 11 tens, instead of which 11 solids of ten, you placed on the table one solid of a hundred and one solid of ten to answer the same purpose. You see, then, that the sum of the 253 and 61 is 3 hundreds, 1 ten, and 4 units, or the same as is expressed by the figures 314, or *three hundred and fourteen*.

[We will hereafter, for the sake of convenience, call the solid of a unit, a *unit*, the solid of ten, a *ten*, the solid of a hundred, a *hundred*, the solid of a thousand, a *thousand*, the solid of ten thousand, a *ten thousand*, the solid of a hundred thousand, a *hundred thousand*.] Let us take another example in *Addition*. Add together 6352, 5648, and 30. You will have no difficulty in expressing these numbers by the solids, as well as in figures, as you see them delineated in Plate V. Figure 2. You add the 8 units to the 2 units and it will make 1 ten, which you "*carry*" to the ten's place; then you add the 3 tens to the 4 tens and it will make 7 tens, and these 7 tens added to the 5 tens will make 12 tens, and 1 ten you "*carry*" from the unit's place will make 13 tens, or 1 hundred and 3 tens, you put down the 3 tens in the ten's place and "*carry*" the 1 hundred to the hundred's place. Then the 6 hundreds and the 3 hundreds make 9 hundreds, and 1 hundred we carry from the ten's place is 10 hundreds, or 1 thousand, which we carry to the thousand's place. Lastly, the 5 thousands added to the 6 thousands make 11 thousands, and the 1 thousand, which we carry from the hundred's place make 12 thousand, or 1 ten thousand and 2 thousand. We then have for the sum of the numbers 6352, 5648, and 30, 1 ten thousand, 2 thousands, *no* hundreds, 3 tens, and *no* units, or 12030, which figures are read, *twelve thousand and thirty*.

If you look at the numbers which I have given you to add together, you will perceive that these numbers are all *different from each other*. Thus the 3 units and the 4 units, which you were required, on page 34, to add together, are not the same. Nor are the 7 units and the 5 units, which are added together in Plate IV. Figure 4. The same may be said of the 56 and 3 in Plate V. Figure 1; of the 253 and 61 in Plate IV. Figures 2 and 3; and likewise of the 6352, the 5648, and the 30 in Plate V. Figure 2. Now if we are required to put together numbers which are *alike*, as, for instance, to add 5 and 5 and 5 and 5 together, we should not say 5 and 5 make 10 and 5 make 15 and 5 make 20: but in order to perform the operation in a shorter way, should say 4 times 5 are 20, which *amounts to the same thing*. If, indeed, we said 5 and 5 make 10 and 5 make 15 and 5 make 20, we should put these four 5s together by ADDITION, which is one

mode of putting numbers together; but if we say four times 5 are 20, we put these four 5s together by

MULTIPLICATION.

Multiplication is the other mode of putting numbers together. You will see in Plate V. Figure 3, the four 5s put together by *Addition.* Now if you set down a 5 on your slate and under it write a 4, or the number of 5s you wish to put together, and say 4 times 5 is 20, and set down the 20 beneath the line drawn under* the 4, you MULTIPLY 5 by 4, or take the 5 *four times.* This *multiplying* one number by another, or taking a number a certain number of times, is what is called MULTIPLICATION. You will see four 5s put together by *Multiplication* in Plate V. Figure 4. You see, that in the putting together of the four 5s by *Addition,* in Plate V. Figure 3, we make no less than *six* figures, while in putting together the four 5s by *Multiplication,* in Plate V. Figure 4, we make only *four* figures; so that by using *Multiplication,* which we can always do when the numbers to be put together are *alike,* the operation is shorter than the mode of putting together numbers by *Addition.* If we had, for instance, to put together the numbers 7, 5, 8, 4 and 6, we could not use *Multiplication* because these numbers are *not alike;* but if we had five 7s, or five 5s, or five 8s, or five 4s, or five 6s, to put together, we should employ *Multiplication* because it is *shorter* than *Addition.*†

* It is sometimes the practice in *Multiplication,* to write the number to be multiplied, which is called the *multiplicand,* and the number by which you multiply, called the *multiplier,* in a *horizontal position,* with an *oblique cross* between them; thus 5×4; and afterwards the *result* of the multiplication, which result is called the *product,* preceded by two horizontal lines, $=$, which means *equal to;* thus $5\times4=20$; that is, 5 multiplied by 4 is equal to 20. Also in *Addition,* particularly when the numbers are small, we place the numbers to be added, in a *horizontal position* with an *upright cross* between them, thus $5+6+4+8$, with the sign of equality between the last number and the sum, thus $5+6+4+8=23$; that is, 5 added to 6 added to 4 added to 8 are equal to 23.

† It is said in some of the treatises on Arithmetic, of deservedly high reputation, that Multiplication is the *same thing* as Addition; but is it not better to consider *Addition* and *Multiplication* as they are regarded in these Instructions, as *different modes of doing the same thing,* that is, of *putting numbers together.*

Let it be required to multiply 256 by 4, that is, to find what two hundreds, five tens, and six units amount to, when multiplied by four units. It will be better to write on the slate the 256 with the 4 under the 6 in the 256, and to place the solids which represent the 256 on the table, as you see in Plate V. Figure 5. Now we say, in the first place, 4 times 6 are 24; but 24 what? 24 units, or 24 tens, or 24 what? If you should say 24 units, which is the right answer, I ask why 24 units, rather than 24 of anything else? To see clearly the reason *why* the 6 units multiplied by 4 units produce 24 units, we must consider, that 1 unit multiplied by 1 unit produces *one unit;* that 1 unit multiplied by 2 units produce 2 units; that 2 units multiplied by 2 units produce 4 units; and that, universally, *units multiplied by units produce units.* Therefore, the 6 units multiplied by the 4 units will make 24 *units;* because, if units be multiplied by units the product will be units. Now instead of counting off 24 unit solids to represent the product of 6 units by 4 units, you may place on the table two solids of ten with four solids of a unit; and we shall have 2 tens and 4 units. We put down the 4 units, and "*carry*" the 2 tens to the ten's place. We then say, 4 times 5 are twenty; but 20 what? To answer this question, *understandingly*, we must consider that 1 ten multiplied by 1 unit produces 1 ten; that 1 ten multiplied by 2 units produces twenty, or two tens; that 2 tens multiplied by 2 units produces 4 tens; and that, universally, *tens multiplied by units produce tens.* The 5 tens multiplied by the 4 units, then, produce 20 tens; because, if tens be multiplied by units, the product will be tens. These 20 tens increased by the two tens which we "*carry*" from the ten's place, will make 22 tens, or two hundred and 2 tens. We set down the two tens and "*carry*" the 2 hundreds to the hundred's place. We then say, 4 times 2 make 8; but 8 what? To answer this, we must consider, that, if we multiply 1 hundred by 1 unit the product will be 1 hundred; that 1 hundred multiplied by 2 units will produce 2 hundreds; that 2 hundreds multiplied by 2 units will produce 4 hundreds; and, universally, *hundreds multiplied by units will produce hundreds.* The 2 *hundreds* multiplied by the 4 *units*, then, will make 8 *hundreds;* because, if hundreds be multiplied by units the product will be hundreds. Then 8 hundreds added to the

2 hundreds, carried from the ten's place, will make 10 hundreds; which will be most conveniently expressed by writing down a 0 in the hundred's place to signify *no* hundreds, and putting 1 in the thousand's place to signify 1 thousand, or represented in the solids by placing on the table a solid of a thousand to represent 1 thousand. If, then, we multiply 256 by 4, the product will be 1 thousand, no hundreds, 2 tens and 4 units, or 1024, which is read *one thousand and twenty-four.*

Take another example in ***Multiplication.*** Multiply 232 by 22.* We are required in this example to multiply 2 hundreds 3 tens and 2 units by 2 tens and 2 units. It will, perhaps, be convenient, to write down on the slate the 232, and to place the 22 under the right hand figures of the 232. We are then directed by the rule in the Arithmetic, to say twice 2 is 4, twice 3 is 6, and twice 2 is 4, setting the product of the 232 by the first 2 in the 22, as this product arises, thus,

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232
 22
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464
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and then to multiply the 232 by the second 2 in the 22, saying twice 2 is 4, and to place this 4 directly under the 2 by which you are multiplying; then to say twice 3 is 6, twice 2 is 4, and to place them down accordingly, thus,

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 232
  22
 ---
 464
464
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After these products are obtained we are directed by the rule, "*to add them up as they stand;*" if this be done, the whole operation will stand thus,

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 232
  22
 ---
 464
464
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5104
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* This example belongs to what the books on Arithmetic term "*long Multiplication,*" because the multiplier exceeds 12. All examples in Multiplication having multipliers of 12, or less than 12, belong to what is technically called "*short Multiplication.*"

And 5104 is the whole product of 232 multiplied by 22. Now every thing about this operation appears plain, except the *reason why* we remove the right hand figure of the product by the second figure of the 22, one place further towards the left. If we say, that this removal is made, "*because the right hand figure of each product must stand directly under that figure of the multiplier by which you multiply*," we offer no satisfactory *reason*. We, therefore, insist on knowing *why* it is, that the first 4 in the second product in this example, is put under the 6, and not under the 4 in the first product, thus,

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232
 22
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464
464
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Neither indeed do we assign an adequate reason for this removal of the right hand figure of the second product one place towards the left, by saying, that in multiplying by the left hand 2 in the 22, we multiply by *tens*. Let us see, then, what the *true reason is* for such a step. For this purpose, you may write the figures while the example is being illustrated by the solids.*

In Plate VI. Figure 1, we have the whole operation of multiplying 232 by 22 written down in figures; and also illustrated by the solids. By attending to the solids, you see that we have two hundreds, three tens, and two units, represented as multiplied by the two tens and two units. In this representation of the solid we begin with the 2 units, and say twice 2 are four *units*, because we multiply units by units; that is, the 2 units in the 232 by the 2 units in the 22, and we have before seen, that, *if units be multiplied by units the products will be units;* Page 39. We put down then the 4 units, and say twice 3 make 6, and 6 *tens*, because we *multiply tens by units*. Placing down the 6 tens, we say twice 2 are 4, and 4 *hundreds*, because we *multiply hundreds by units*. We have thus obtained the product of 2 hundreds, 3

* It will be best, whether we use the slate or the black board, to *copy* with the figures *simultaneously* with illustrating the operations, and the principles on which they depend, by means of the solids. The pupil will thus pursue the most effectual course to acquire a competent knowledge, and a rational idea of the *import of figures*, and the *meaning of their language*.

tens, and 2 units by the 2 units in the 22, and find it to be 4 hundreds, 6 tens, and 4 units. We must now find the product of the 2 hundred, 3 tens, and 2 units by the 2 *tens* in the 22. To do this, we multiply by the 2 tens, and say twice 2 are 4; but 4 what? To answer this question, we must consider that, if we multiply 1 unit by 1 ten, the product will be 1 ten; that, if we multiply 1 unit by 2 tens the product will be 2 tens; that, if we multiply 2 units by 2 tens, the product will be 4 tens; and, that, universally, *if we multiply units by tens, the product will be tens.* So that when, in this example, we say twice 2 are 4, it is 4 tens, because we *multiply units by tens.* To express these 4 tens, we put down the 4 tens in solids on the table as directed in the Plate; and since these 4 tens form a part of the product of the 232 by the 2 tens in the 22, we may as well put them in the most convenient place under the product of the 232 by the 2 *units* in the 22, to add them to this first product, after the multiplication shall have been performed. We, accordingly, place these 4 tens directly under the 6 tens in the first product, which is the same thing as *placing the first figure of the second product one remove towards the left.* We now proceed with our multiplication of the 232 by the 2 tens in the 22, and say, twice 3 are 6; but 6 what? To answer this question, we must consider that, if we multiply 1 ten by 1 ten the product will be 1 hundred;* that if we multiply 1 ten by 2 tens, the product will be 2 hundreds; that, if we multiply 2 tens by 2 tens, the product will be 4 hundreds; and, that, universally, *if we multiply tens by tens, the product will be hundreds.* When, therefore, in the present instance, we say twice 3 will make 6, the 6 will be 6 hundreds, because, if we multiply tens by tens the product will be hundreds. We, lastly, say twice 2 are 4; but 4 what? To determine this,

* In our multiplication of tens by units, tens by tens, &c., there is nothing analagous to the "Questio quondam vexata" of the "*smashers in cyphering*" concerning the multiplication of *two and six pence by two and six pence*. It is perfectly evident, that we can as well multiply two horses by two horses as two shillings by two shillings; and it is equally clear, that we can no more multiply two shillings by six pence, than we can multiply two dogs by six cats. The simple reason of this is, that we cannot multiply *concretes* by *concretes*. But since we can multiply *abstracts* by *abstracts*, and since one *ten* is as much an abstraction of the mind as one *unit*, it is as proper to multiply tens by tens and units by tens, as it is to multiply units by units; and the same is true, from the nature of the subject, of all other abstractions whatever.

we must consider, that if we multiply 1 hundred by 1 ten, the product will be 1 thousand; that, if we multiply 1 hundred by 2 tens, the product will be 2 thousands; that, if we multiply 2 hundreds by 2 tens, the product will be 4 thousands; and, that, universally, *if we multiply hundreds by tens, the product will be thousands.* Our 4, then, will be 4 thousands, because we multiply hundreds by tens. We put down the 4 thousands, and thus complete the product of the 232 by the 2 tens in the 22, and find it to amount to 4 thousands 6 hundreds and 4 tens. To obtain, then, the whole product of the 232 by the 22, we add together the product of the 232 by the 2 *units* in the 22, and the product of the 232 by the 2 *tens* in the 22. By this addition, we have, first, 4 units; then, the 6 tens in the first product and the 4 tens in the second product will make 10 tens, instead of which we put down 1 hundred, which is the same thing as 10 tens, to be carried to the hundred's place. We then add the 6 hundreds in the second product, to the 4 hundreds in the first product, and their sum will be 10 hundreds, and 1 hundred to be carried from the ten's place, will make 11 hundreds. We substitute, for 10 of these hundreds, 1 thousand to be carried to the thousand's place. We, lastly, have 4 thousands, and 1 thousand to carry from the hundred's place which will make 5 thousands. We, then, have for the product of 2 hundreds 3 tens and 2 units by 2 tens and 2 units, 5 thousands, 1 hundred, *no* tens, and 4 units; or 5104, which is read, *five thousand one hundred and four.*—You perceive, by this time, that, in all cases of multiplication, in order to determine what the product will be, you must inquire *what* you multiply, and *by* what you multiply. If, for instance, as in the example we have just performed, you say twice 2 is 4, you must find, in the first place, whether this 2 which you multiply is 2 units, or 2 something else. If you see that it is two units, you must, in the second place ascertain what the '*twice*,' or the 2 *by* which you multiply the 2 units, whether it is 2 units, 2 tens, or 2 something else; if it is 2 units, then your twice 2, *because you multiply units by units*, will make 4 units. But, if the 2 *by* which you multiply is 2 tens, then your "twice 2 are 4," will make 4 tens, *because you multiply units by tens*. So that if you thus *reason* on the subject, as you proceed, you will comprehend the *principles*, of the whole operation. You may have

observed, that when we have one number to be multiplied by another, we call the number which we multiply, the *multiplicand;* the number *by* which we multiply, the *multiplier;* and the number which is *produced* by the multiplication, the *product.* Thus if we multiply the number 232 by 22, the multiplication will, as you have seen, produce 5104. Now in this multiplication we call the 232 the *multiplicand;* the 22, the *multiplier;* and the 5104, the *product.* As we shall occasionally use these terms, *multiplicand, multiplier,* and *product,* you will recollect their meaning.

There are three "*cases*" of multiplication, as they are named in the Arithmetics, which commonly puzzle learners; and which we shall make you understand. The *first* of these *cases* is, where you have cyphers "*annexed*"* to the multiplier; the *second* is where cyphers are "*annexed*" to the multiplicand, and the *third* is, where cyphers are "*annexed*" both to the multiplier and multiplicand. We will explain these three cases in the order, in which we have mentioned them. And first, we will take the case in which cyphers are annexed to the multiplier. Let it be required, then, to multiply 232 by 220. Here we have a cypher annexed to the multiplier. That is, 2 hundreds, 3 tens, and 2 units for our multiplicand, and 2 hundreds and 2 tens for our multiplier; and to this multiplier of 2 hundreds and 2 tens, there is, and there must be, when we write it in figures, a cypher annexed, to show that the 22 in this multiplier *means* 2 hundreds and 2 tens, or 22 tens; for, if there were no cypher annexed to this 22, it would mean 22 units.†

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* It will be well for the pupil clearly to understand the distinctive meaning of these three terms, viz. *annex, prefix,* and *add;* since the confounding of their signification frequently leads to obscurity. To *annex,* then, means to put or place *after;* to *prefix,* to place *before,* and to *add,* to *incorporate* with. If we annex 2 to 4, it will make 42; if we prefix 2 to 4, it will make 24; and if we *add* 2 to 4, it will make 6.

† There is, indeed, a metaphysical impossibility involved in the expression of the books, "cyphers annexed to a number," as numbers are written in *figures.* The number 220 is 220, and no more. If we annex a cypher to this number it becomes 2200. No number, then, strictly speaking, can have cyphers *annexed* to it, and remain what it is. A number may *consist* partly *of,* or be *expressed* partly *by,* cyphers; but the phrase, "cyphers *annexed to a multiplier, multiplicand, or product,*" or any other number, if we suppose the number to remain, unalterably, what it is, implies an absurdity. These remarks may be deemed hypercritical by many, but any one, who has had some "*taste,*" as Lord Coke would say, of the difficulty of imparting to beginners clear ideas on the subject of Arithmetic, will appreciate their intention, and feel their weight.

You may, now, set down your multiplicand and multiplier on the slate, and perform the illustration by means of the solids. See Plate VII. You put down, then, the 2 hundreds and 3 tens and 2 units, for the multiplicand, and the 2 hundreds and 2 tens, for the multiplier. Begin the multiplication by saying twice 2 are 4; and this 4 will be 4 *tens*, because you multiply *units by tens*: Page 42. Then twice 3 are 6; and this six will be 6 *hundreds*, because you multiply *tens by tens*. Then twice 2 make 4; and this 4 will be 4 *thousand*, because you multiply *hundreds by tens*. You have now 4 thousands 6 hundreds and 4 tens, for the *first* product, or the product of 232 by the 2 tens in the 220. We now multiply the multiplicand by the 2 hundreds in the 220, and say twice 2 make 4; but 4 what? To determine this, we must consider, that if we multiply 1 unit by 1 hundred, the product will be 1 hundred; that, if we multiply 1 unit by 2 hundreds, the product will be 2 hundreds; that, if we multiply 2 units by 2 hundreds, the product will be 4 hundreds; and, that, universally, *if we multiply units by hundreds, the product will be hundreds.* The 4, therefore, will be 4 hundreds, because we multiply units by hundreds. We, then, say twice 3 will be 6; but 6 what? To ascertain this, we must consider, that, if we multiply 1 ten by 1 hundred, the product will be 1 thousand; that, if we multiply 1 ten by 2 hundreds, the product will be 2 thousand; that, if we multiply 2 tens by 2 hundreds, the product will be 4 thousands; and, that, universally, *if we multiply tens by hundreds, the product will be thousands.* Our 6, therefore, is 6 thousands. We, then, say twice 2 are 4; but 4 what? To decide this, we must consider, that, if we multiply 1 hundred by 1 hundred, the product will be 1 ten thousand; that, if we multiply 1 hundred by 2 hundreds, the product will be 2 ten thousands; that, if we multiply 2 hundreds by 2 hundreds, the product will be 4 ten thousands; and, that, universally, *if we multiply hundreds by hundreds, the product will be ten thousands.* Our 4, therefore, will be 4 ten thousands. You have now 4 ten thousands 6 thousands and 4 hundreds, for the *second* product, or the product of the 232 by the 2 hundreds in the 220. We will add the first and second products together, in order to obtain the whole product of 232 by 220. And, first, we have 4 tens; then, 4 hundreds and 6 hundreds make 10 hundreds, for which we sub-

stitute 1 thousand to be carried to the thousand's place; then, 6 thousands and 4 thousands make 10 thousands and 1 thousand we carry from the hundred's place, makes 11 thousands. For 10 of these thousands, we substitute 1 ten thousand to carry to the ten thousand's place, and leave the 1 thousand in the thousand's place. Lastly, we have 4 ten thousands and 1 ten thousand, which we carry from the thousand's place makes 5 ten thousands. We thus have for our product of 232 by 2 hundreds and 2 tens, 5 ten thousands 1 thousand and 4 tens, or 51040. Now if you examine the *copy* of this operation in figures, you will see that at the moment you have completed the process of multiplication by the two 2s in the multiplier, and *before* you have brought down the cyphers "*annexed*" to the multiplier,

```
   232
    220
  -----
   464
  464
  -----
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your product will stand in the figures thus, 5104 And you are directed by the Rule laid down in the books, for this "*case*" of Multiplication, that, "*when there are cyphers at the right hand of the multiplier, you must perform the operation with the figures; and, neglecting these cyphers until you have found the product by the figures, annex as many cyphers to this product by the figures, as there are cyphers at the right hand of the multiplier.*" In some of the books this is called "*bringing down*" cyphers to the right of the product. You see, then, the *reason* of this direction in the Rule, to "*bring down*" cyphers to the right hand of the product of the multiplicand by the "*figures*" in the multiplier; for the product of 232 by 2 hundreds and 2 tens, when read at length, will be 5 ten thousands 1 thousand *no* hundreds 4 tens and *no* units; and in figures, when there is nothing in any place we must put a cypher there to signify that there is *nothing* in that place. You, therefore, annex a cypher to the 5104 mentioned above to signify *no* units; and the 5104, which is the product in *figures* before the cypher is "*brought down,*" will stand 51040, and be read *fifty-one thousand and forty.*

We will now illustrate the *second case;* or where cyphers are

annexed to the multiplicand; and for this purpose multiply 2320 by 22. Set down the figures, and place the solids to illustrate their meaning as in the preceding examples. See Plate VIII. We here have 2 thousands 3 hundreds and 2 tens to multiply by 2 tens and 2 units; and we say twice 2 are 4; and this 4 is 4 tens, because we multiply tens by units. Then twice 3 are 6; and this 6 is 6 hundreds, because we multiply hundreds by units; then twice 2 are 4; but 4 what? To determine this, we must consider, that, if we multiply 1 thousand by 1 unit, the product will be 1 thousand; that, if we multiply 1 thousand by 2 units, the product will be 2 thousands; that, if we multiply 2 thousands by 2 units, the product will be 4 thousands; and that, universally, *if we multiply thousands by units, the product will be thousands.* Our 4, therefore, will be 4 thousands, because we multiply thousands by units. We thus have 4 thousands 6 hundreds 4 tens and *no* units for the *first* product, or the product of 2 thousands 3 hundreds and 2 tens by 2 units. To obtain our *second* product, or the product of 2320 by the 2 tens in the 22, we say, twice two are 4; and this 4 is 4 hundreds, because, if we multiply tens by tens, the product will be hundreds. Then, twice 3 are 6; and this 6 is 6 thousands, because if we multiply hundreds by tens, the product will be thousands. Lastly, twice 2 are 4; but 4 what? To answer this question, we must consider, that, if we multiply 1 thousand by 1 ten, the product will be 1 ten thousand; that, if we multiply 1 thousand by 2 tens, the product will be 2 ten thousands; that, if we multiply 2 thousands by 2 tens, the product will be 4 ten thousands; and, that, universally, *if we multiply thousands by tens, the product will be ten thousands.* Our 4, therefore, will be 4 ten thousands, because if we multiply thousands by tens, the product will be ten thousands. We then have, for our *second* product, or the product of 2 thousands 3 hundreds and 2 tens by 2 tens, 4 ten thousands 6 thousands and 4 hundreds. We now add these two products together in order to obtain the whole product of 2320 multiplied by 22. First, then, we have 4 tens. Then, 4 hundreds and 6 hundreds are 10 hundreds, instead of which we put down 1 thousand to be carried to the thousand's place. Then 6 thousands and 4 thousands are 10 thousands, and 1 thousand we carry from the thousand's place, makes 11 thousands. For 10

of these thousands we put down 1 ten thousand to be carried to the ten thousand's place, and leave the 1 thousand in the thousand's place. Lastly, we say 4 ten thousands and 1 ten thousand we carry from the thousand's place makes 5 ten thousands. And we have 5 ten thousands 1 thousand and 4 tens, for our product of 2320 by 22. If we attend particularly to this example of multiplication as written in figures, we shall find, that, *after* the multiplication of the *figures only* in the multiplicand by the multiplier is performed, and *before* the cypher in the multiplicand is "brought down to the right hand of the product," the example will stand thus,

```
  2320
    22
  ----
   464
  464
  ----
  5104
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5104 or 5 ten thousands 1 thousand *no* hundreds 4 tens and *no* units. Now we must have a cypher in the units place to EXPRESS *no* units; and to set down this cypher at the right hand of the 5104,* would be the same thing as to "*annex as many cyphers to the product of the figures in the multiplicand by the multiplier, as there are cyphers at the right hand of the multiplicand.*" If, then, we annex this cypher, our 5104 will be read 51040, the correct and *entire* expression in figures of the product of 2320 multiplied by 22.

The third and last case of multiplication is that in which cyphers are annexed to both multiplier and multiplicand, at the same time. To acquire a clear idea of the principle involved in this case, you may multiply 230 by 220. See Plate IX. We have, in this example, 2 hundreds and 3 tens to be multiplied by 2 hundreds and 2 tens. We say twice 3 are 6, and this 6 is 6 *hundreds*, because we multiply tens by tens; then, twice 2 are 4, and this 4 will be 4 *thousands*, because we multiply hundreds by tens. We thus have 4 thousands and 6 hundreds for our product of the 230 by the 2 tens in the 220. We then say, twice 3 are

* By a little reflection on this and the last example, the pupil will perceive that these figures, 5104, actually mean five thousand one hundred and four. But their meaning is 5104 tens—five thousand one hundred and four *tens*. And this 5104 tens is the same thing as 51040 units.

6; but 6 what? To determine this, we must consider, that, if we multiply 1 ten by 1 hundred, the product will be 1 thousand; that, if we multiply 1 ten by 2 hundreds, the product will be 2 thousands; that, if we multiply 2 tens by 2 hundreds, the product will be 4 thousands; and, that, universally, *if we multiply tens by hundreds, the product will be thousands.* Our 6, therefore, will be 6 *thousands*, because we multiply *tens* by *hundreds.* We then say, twice 2 make 4; but 4 what? To determine this, we must consider, that, if we multiply 1 hundred by 1 hundred, the product will be 1 ten thousand; that, if we multiply 1 hundred by 2 hundreds, the product will be 2 ten thousands; that, if we multiply 2 hundreds by 2 hundreds, the product will be 4 ten thousands; and, that, universally, *if we multiply hundreds by hundreds, the product will be ten thousands.* Our 4, therefore, will be 4 *ten thousands*, because we multiply *hundreds* by *hundreds.* We have, then, for the product of the 230 by the 2 hundreds in the 220, 4 ten thousands and 6 thousands, or 46 thousands, which, if we add it to the 4 thousands and 6 hundreds (our first product) will make 5 ten thousands *no* thousands 6 hundreds *no* tens and *no* units. Now to express this number in figures, after we have written the 506 as the whole product of the *figures* will stand before any cyphers are brought down,

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 230
 220
 ───
  46
 46
 ───
 506
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we must annex to this 506 *two* cyphers, or as many cyphers as there are at the right of both the multiplier and multiplicand, to mean *no tens* and *no units.* The 506* will become 50600, or fifty thousand six hundred, which is the product of 230 multiplied by 220. In these examples of Multiplication, in which cyphers are annexed either to multiplier or to multiplicand, or to both, there has been only *one* cypher annexed in each case. We will give you one more example to show, that, if the annexed cyphers are *more* than

* By a little consideration, the pupil will perceive, that this 506 may actually be read, *five hundred and six;* but it will 506 *hundreds—five hundred and six hundreds;* and, that this five hundred and six hundreds is the same thing, in fact, as *fifty thousand six hundred*, that is, 5 ten thousands and 6 hundreds are equal to fifty thousand and six hundred *units.*

one, as many cyphers must be set down at the right hand of the product, after the operation is performed by the figures separately, as there are "*cyphers annexed* to the multiplier and multiplicand." You may multiply 230 by 200, that is, 2 hundreds and 3 tens by 2 hundreds. See Plate X. Figure 1. In this example, we say twice 3 are 6; and this 6 is 6 *thousands*, because we multiply *tens* by *hundreds*. Then twice 2 are 4; and this 4 is 4 ten thousands, because we multiply hundreds by hundreds. We thus have obtained 4 ten thousands and 6 thousands, for our product of 230 by 200. If we set this example down in figures, before we set down the cyphers annexed, it will appear thus,

$$\begin{array}{r} 230 \\ 200 \\ \hline 46 \end{array}$$

To this 46,* which means 4 ten thousands and 6 thousands, or 46 thousands, we must annex 3 cyphers, or as many cyphers as there are annexed both to multiplier and multiplicand, to make it read as it properly should, 46000, four ten thousands 6 thousands *no* hundreds *no* tens and *no* units, or *forty-six thousands*.

We have thus given you an adequate idea of the principles of Multiplication, or the second mode of *putting numbers together*. There are also *two* modes of separating numbers from each other, or of *taking numbers apart*. And you will remember, that, in every operation in Arithmetic, which is the *science of numbers*, there is only one of two things to be done, viz. either TO PUT NUMBERS TOGETHER, or, TO TAKE THEM APART. Since you now understand the two methods of *putting numbers together*, we shall next proceed to make you understand the two methods of *taking numbers apart*. And these two methods of taking numbers apart, are called SUBTRACTION and DIVISION. We will first give you a clear idea of

SUBTRACTION.

If it were required to take 5 from 9, it is perfectly plain that 4 would remain. There is not the slightest difficulty, indeed, in

* The pupil will perceive, by a little attention, that this 46 is actually *forty-six*; but 46 *thousands*, that is, that 4 ten thousands and 6 thousands are the same thing as forty-six thousand units, or, as it is *customarily* read, *forty-six thousand*.

taking *one* figure, or rather the number *expressed* by *one* figure, from another number expressed by *one* figure: as 5 from 9, 3 from 7, 4 from 6, &c. It is only when you have numbers *expressed by several figures* to take from other numbers expressed by several figures, as 235 from 346, that you meet with any difficulty in Subtraction. Still greater does this difficulty become, when you are required to take a *larger* figure in the number to be subtracted, from a *smaller* figure in the number *from which* you subtract. For instance, if you were required to subtract 61 from 253, you would set them down thus,

$$\begin{array}{r} 253 \\ 61 \\ \hline 192 \end{array}$$

with the larger number, which is called the *minuend*, over the smaller number, which is called the *subtrahend*. You would then say 1 from 3 leaves 2; then 6 from 5 I cannot, but 6 from 15 leaves 9. Lastly, 1 *to carry* to nothing is 1, which subtracted from 2 leaves 1. 192 then, is what remains after 61 is subtracted from 253, and is called the *remainder*. Now it is this "*borrowing and carrying*" in Subtraction, which puzzles the learner of Arithmetic more, if possible, than does the *carrying* in Addition. And to cause you fully to comprehend the *reason why* you "borrow and carry" in Subtraction, we shall give you a few examples.

Example 1. Let it be required to subtract 61 from 253. See Plate VI. Figure 3. We have here to take 6 tens and 1 unit from 2 hundreds 5 tens and 3 units. It will be most convenient to take units from units, tens from tens, &c., and this is the reason *why*, in setting down our *minuend* and *subtrahend*, we are directed by the Rule in the books on Arithmetic, *to place units under units, tens under tens, &c.* We say, then, in the present example, 1 unit from 3 units will leave 2 units, and put these 2 units down in solids. We then say 6 tens from 5 tens; and it is here that the difficulty occurs. Supposing, then, we take from 253 one of the hundreds, and put 10 tens instead of it, by the side of the other tens in the 253, as in Plate VI. Figure 4. We then have 1 hundred 15 tens and 3 units, the same thing as 2 hundreds 5 tens and 3 units, or as 253. Now you can take the

6 tens from these 15 tens, and there will remain 9 tens. There is nothing, then, to subtract from the 1 hundred, which remains entire, for which we put down another hundred, and the remainder is 1 hundred 9 tens and 2 units, or 192.

You will now be prepared to understand the operation of subtracting 61 from 253. See Plate VI. Figures 3 and 4. You say 1 from 3 leaves 2 units;—6 from 5 I cannot, but taking 1 of the 2 hundreds in 253 and changing it to 10 tens, I have 1 hundred and 15 tens, (adding the 5 tens to the 10 tens) and 3 units equal to 253. I then say 6 from 15 leaves 9 tens; and say, lastly, nothing from 1 leaves 1 hundred; for the 2 hundred has become 1 hundred, in consequence of 1 of the 2 hundreds being taken away, and 10 tens put in its place. You see, then, that this "*borrowing and carrying*" amounts to nothing more, than "*imagining*" 1 to be taken from the next higher place, where a figure in the subtrahend is greater than the figure from which it is to be taken in the minuend, and this 1 to be reduced to 10 of the lower place, and then 1 to be returned to the next lower figure in this next higher place, to compensate for what was imagined to be taken away from the upper figure. By performing the operation with the solids, we see what does, *in fact*, take place. Throughout the operation by figures, every thing is *imaginary*. If we employ the solids, we see that the 2 hundreds and 5 tens and 3 units, or the 253, must and *does*, if we wish to show what *actually* occurs, become 1 hundred and 15 tens and 3 units; that, after having taken the 1 unit in the 61 from the 3 units in the 253, we say 6 tens from the 15 tens leaves 9 tens; and that, then there is nothing to be taken from the 1 hundred, which remains *entire*. Now if we set down the figures thus,

$$\begin{array}{r} 253 \\ 61 \\ \hline 192 \end{array}$$

we say 1 from 3 and 2 remains; then 6 from 5 we cannot, but "*imagining*" 1 hundred to be taken from the 2 hundreds, reduced to 10 tens and added to the 5 tens making 15 tens, we say 6 from 15 leaves 9. Then because the 2 hundreds, as expressed by the figure 2 *actually* remains *unchanged*, although we "*imagine*" it to be changed to 10 tens, we must say 1 to carry

to nothing is 1, and this 1, which is thus carried, taken from the 2 leaves 1.

Example 2. Let it be required to subtract 1 from 300. We should, according to the direction in the *Rule* of the Books, set down the figures of this example as you see in Plate X. Figure 2. We then say 1 from nothing we cannot, but 1 from 10 leaves 9; but we ask 1 *what* from 10 *what?* The answer probably would be, 1 *unit* from 10 *units.* The answer is correct; but it may be asked, whence do we obtain the 10 units? The answer almost universally given by the pupil is, "*I 'borrow'* 1 *from the ten's place, and reduce it to* 10 *units.*" But it may be replied, that is impossible to borrow from the ten's place in this example, because in the 300 there is nothing *in* the ten's place. The pupil then, perhaps, will say, that he will borrow from the next place. But this next place is filled by 3 *hundreds,* and not 3 *tens* nor 3 *units.* Now we want 1 *ten.* To go to the hundred's place, therefore, for our 10 units is going too far. Indeed, there is somewhat of obscurity in this example and other examples involving the same principles with it. If, however, you look at the illustration which you can make with the solids as delineated in Plate X., Figures 2, 3, and 4, you will have little difficulty in understanding the whole subject. You have in Plate X. Figure 2, 3 solids of a hundred put down, and at their right a little lower down, 1 solid of a unit. Now the object of the operation is, to take *this* 1 *unit from the* 3 *hundreds.* We may say, 1 from nothing we cannot. But if we take away one of the 3 hundreds, as represented in Plate X. Figure 2, and place in their room on the table 10 tens, we shall have, as represented in Plate X. Figure 3, 2 hundreds and 10 tens, which are the same thing as 3 hundreds. Still we have nothing to take our 1 unit from. Take away then, 1 of the 10 tens, as represented in Plate X. Figure 3, (leaving 9 tens) and put in its place 10 units as represented in Plate X. Figure 4, and we shall have 2 hundreds 9 tens and 10 units, which are the same thing as 3 hundreds. We may now take our 1 unit from the 10 units, and there will remain 9 units and the 9 units joined to 9 tens and 2 hundreds, which remain entire, will constitute the 2 hundreds, 9 tens and 9 units or 299 which remains where 1 is taken from 300. Now if you recur to the operation as it is expressed by the figures, you will not fail to comprehend

their meaning, and this subject of "*borrowing and carrying*," in subtraction. After having written the figures thus

```
300
  1
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on the slate, you will say one from 0 I cannot take; but 1 unit from 10 units leaves 9 units; and these 10 units are obtained by taking 1 of the 3 hundreds, leaving 2 hundreds, and changing this hundred into 10 tens, and then by taking 1 of the 10 tens, leaving 9 tens, and converting this 1 ten into 10 units. Our 3 hundred will then stand thus, 29(10), or 2 hundred 9 tens, and 10 units, or, to use a phrase somewhat *unusual, two hundred and ninety-ten,* and this expression conveys to the mind the same idea with the phrase *three hundred.* Then our example may thus be expressed in figures, as you will find it in Plate X. Figure 4,

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29(10)
300
  1
------
299
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In this analysis the 3 in the hundred's place becomes 2, the cypher in the ten's place, becomes 9, and the cypher in the unit's place becomes 10. Then we say 1 from 10 leaves 9, nothing from 9 leaves 9, and nothing from 2 leaves 2. * †

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* The *ancient* method to solve this subtraction of 1 from 300 is this,

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300
  1
---
299
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saying "1 *from nothing we cannot, but* 1 *from* 10 *leaves* 9; 1 *to carry to nothing is* 1, *and from nothing we cannot, but from* 10 *leaves* 9; 1 *to carry to nothing is* 1, *and from* 3 *leaves* 2."

† In the ordinary *descriptive* process of the books, the figures are suffered to remain as they are. All the changes which take place in them are entirely *imaginary*. According to this descriptive process we say 1 from 0 we cannot take, but *imagining* 10 to be added to the unit's place, we say, 1 from 10 leaves 9; then *imagining* 1 to be put in the ten's place of the subtrahend, to compensate for its equal 10 units, which are added to the unit's place of the minuend, on the principle, that the subtrahend must be increased as much as the minuend, in order to obtain the true remainder, we say 1 from 9 we cannot take, but *imagining* 10 to be added to the ten's place of the minuend, we say, 1 from 10 will leave 9, then we put 1 in the hundred's place of the subtrahend to compensate the 10 tens we added to the ten's place in the minuend, on the principle of the equal increase of minuend and subtrahend just stated, and say, 1 from 3 leaves 2. We thus obtain our whole remainder 299.

Example 3. From 2000 take 24. In this example, we are required to subtract 2 tens and 4 units from 2 thousands. You will see the operation expressed in figures and illustrated by the solids in Plate X. Figure 5. We here say, 4 units from *no* units we cannot take, but 4 units from 10 units leave 6 units; and the manner in which we obtain our 10 units, you will clearly comprehend, by examining the change, which actually takes place in the solids or in transferring the illustration made by them, from Plate X. Figure 5, to Figure 6 of Plate X. In Figure 7 of Plate X. you will see the various changes, which successively occur in the minuend 2000, previously to its transfer from the condition in which it exists in Plate X. Figure 5, to that at which it arrives in Plate X. Figure 6. You say in Plate X. Figure 5, 4 units from nothing I cannot subtract. But on looking at Figure 7, you will see, that if you take away 1 of the 2 thousands, and substitute for it, 10 hundreds, you will have one thousand and 10 hundreds, equal to 2 thousands. Then if you take away 1 of these hundreds, leaving 9 hundreds, and place in its stead, 10 tens, you will have 1 thousand, 9 hundreds, and 10 tens, equal to 2 thousands. Lastly, if you take away 1 of the 10 tens leaving 9 tens, and change this 1 ten to ten units, you will have 1 thousand 9 hundreds, 9 tens, and 10 units as in Plate X. Figure 6, equal to 2 thousands. Now you can say, 4 units from 10 units leaves 6 units, then 2 tens from 9 tens leave 7 tens. Then *no* hundreds from 9 hundreds leaves 9 hundreds. Lastly, *no* thousands from 1 thousand leaves 1 thousand. You thus have 1 thousand, 9 hundreds, 7 tens, and 6 units, for the difference between 2 thousands, and 24 units, or 2 tens and 4 units. If you examine the *minuend* and *subtrahend*, as they are expressed by *figures*, Plate X. in Figure 6, you say, 4 from 0 you cannot take; but 4 units from 10 units leave 6 units. Now to obtain these 10 units, you see clearly, that you take, in the first place, one of the 2 thousands, leaving 1 thousand, and reduce this one thousand to 10 hundreds, then take one of these 10 hundreds, leaving 9 hundreds, and reduce it to 10 tens; then take one of these 10 tens, leaving 9 tens, and reduce it to ten units. Then we have, as expressed above, the 2000, in figures, 1 *thousand* 9 *hundred and ninety-ten*, equal to 2 thousand. Then you can say, 4 from 10 leaves 6; 2 from 9 leaves 7; nothing from 9 leaves 9; and

nothing from 1 leaves 1. Thus we have 24 subtracted from 2000, leaving, for a *remainder*, 1976. You may now perform with the figures, and illustrate with the solids, the following examples.

Example 4. From 10000, subtract 1.

Example 5. From 5256, subtract 4166.

Example 6. From 100000, subtract 3.

Example 7. From 10, subtract 9.

You have by this time acquired an adequate idea of the mode of taking ONE number from ANOTHER number, which mode of *taking numbers apart* is called SUBTRACTION. Now, supposing you should be asked *how many times* you could take 5 from 20, you might, as you see in Plate XI. Figure 1, set down the 20, and the 5 under it, and say 5 from 20 leaves 15; then setting the 5 under the 15, you would say 5 from 15 leaves 10; then setting the 5 under the 10, you would say 5 from 10 leaves 5; then setting the 5 under the 5, you would say 5 from 5 leaves nothing. You thus see, that 5 can be taken from 20 FOUR TIMES. But in thus performing this operation of finding how many times 5 can be taken from 20, we are obliged to make *twelve* figures. Now we may ascertain how many times 5 can be taken from 20, without making more than *four* figures. For this purpose, as you see in Figure 2, Plate XI., you may set down the 20, and put the 5 at its left, drawing a line between them to keep them apart. *Now a less number is contained in a greater just as many times as the less can be taken from the greater.* Thus, 5 is contained in 20 four times, and 5 can be taken from 20 four times. If then after having set down the 20 with the 5 at the left, we ask *how many times* is 5 CONTAINED IN 20, and say 4 times, setting down the 4 at the right of the 20, or under the 20, and drawing a line to keep the 20 and the 4 apart, we actually set down the number of times which 5 can be *taken from* 20. To inquire, then, how many times 5 IS CONTAINED IN 20 amounts to the same thing as to inquire how many times 5 can be TAKEN FROM 20. Again, if after setting down our 20 and 5, as in Figure 2, Plate XI., we inquire, If 20 be divided into 5 equal parts, what will each of these five parts be? the answer will be, there will be 4 in each of these parts. So that, whether we inquire How many times 5 can be TAKEN FROM 20, or How many times 5 IS CONTAINED IN

20, or, What will each of the parts of 20 be, if it is divided into 5 parts, the answer will be 4, or precisely the same thing.* Whenever, therefore, we wish to ascertain *how many times* one number can be *taken from* another, we inquire, at once, how many times the smaller number IS CONTAINED in the larger, or, if the *larger* number be *divided* into as many parts as are signified by the *smaller*, what will each of these parts amount to? We thus, as in Figure 2, Plate XI., obtain the same answer by a *shorter method* than we should if we went through the longer process, as in Figure 1, Plate XI. This shorter method of *taking numbers apart* is called *Division;* because by this method we DIVIDE *the larger number into as many parts as are signified by the smaller number.* Having gone through, then, with the *first* mode of taking numbers apart, or SUBTRACTION, we come to the *second* mode of taking numbers apart, or

DIVISION.

SUBTRACTION, as you have seen, consists simply *in taking a smaller number from a larger number, and thereby showing the difference between them.* DIVISION *is the dividing of a* LARGER *number into as many parts as are denoted by a* SMALLER *number, and thereby ascertaining what each of these parts will be.* The larger number in *Division,* or the number *to be divided,* is called the *dividend;* the smaller number, or the number by which we divide, is called the *divisor;* and the number which denotes the *number of times* the dividend contains the divisor, is called the *quotient.* Sometimes, when we divide one number by another,

* If I mistake not, there has been a distinction made, in some of the books, between those cases of Division, in which the inquiry is, How often the *less* number is contained in the *greater;* and those, in which the object of inquiry is, to ascertain the amount of one of the parts, when the larger number is divided into as many parts as are denoted by the smaller. I can see no reason in making this distinction, unless it is to show, by *varying* the *expression,* that both inquiries are the same. To my own mind, there is, *metaphysically,* so far as number is concerned, the same idea conveyed by the inquiry in question, in whichsoever of the three following forms the interrogation is expressed:

How many times can 5 be taken from 20?

How many times are 5 contained in 20? or,

What will each of the parts of 20 amount to, supposing you divide 20 into 5 equal parts?

there is something left after the division is performed. Thus, if we divide 23 by 5, we see that 5 is contained in 23 four times and 3 over. This number 3, which is *left* after the division is performed, is called the *remainder*. If, therefore, we divide 23 by 5, the answer will be 4 and 3 over. The 23 is the *dividend;* the 5 is the *divisor;* the 4 is the *quotient;* and the 3 is the *remainder*. In the books on Arithmetic, you generally find *two kinds* of Division described; one kind being termed *Short* Division, which takes place when the Divisor is 12, or less than 12; the other kind being called *Long* Division, which occurs when the Divisor is more than 12. Though there is no great difficulty in understanding *Short* Division, we shall give you a few examples of it.

Example 1. Divide 136 by 4. By looking at Plate XI. Figure 3, you will see this example expressed by the *figures* and by the solids. In this example we have 1 hundred 3 tens and 6 units to be divided by 4 units; or, we are to ascertain how many times 4 units are contained in 1 hundred 3 tens and 6 units; or, which is the same thing, to divide 1 hundred 3 tens and 6 units into 4 equal parts, and to find what each of those parts will be. And, first, it will not be convenient to divide the 1 hundred into 4 equal parts, so as to express it by figures, unless we first convert it into 10 tens, which are equal to the 1 hundred. If, therefore, we convert our 1 hundred into 10 tens, we may add to these 10 tens the 3 tens, already in the ten's place, and they will make 13 tens. Now, if we divide 13 tens into 4 equal parts, there will be 3 tens in each of these parts, and 1 ten over, undivided. We put down our 3 tens, and reducing this 1 ten which is over and undivided to 10 units, and add to these 10 units the 6 units, already in the unit's place, they will make 16 units. If we divide these 16 units into 4 equal parts, there will be 4 units in each of these parts; we put down 4 units; and we have 3 tens and 4 units for each of the parts, if we divide 1 hundred and 3 tens and 6 units, into 4 equal parts; that is, if we divide 136 by 4, 34 will be the quotient.

Example 2. Divide 2354 by 11. We are required, in this example, to divide 2 thousands 3 hundreds 5 tens and 4 units into 11 equal parts; or to find how many times 11 units are contained in 2 thousands 3 hundreds 5 tens and 4 units. By

referring to Plate XI. Figure 4, you will see the example set down in figures, and its illustration by the solids delineated. Here we cannot divide the 2 thousands into 11 equal parts.* But by making the 2 thousands 20 hundreds, and by adding to these 20 hundreds, the 3 hundreds already in the hundred's place, we have 23 hundreds. Now, if 23 hundreds be divided into 11 equal parts, there will be 2 hundreds in each of these parts, and 1 hundred over, undivided. If we put down the 2 hundreds, and reduce this 1 hundred, which is undivided, to 10 tens, and add the 5 tens already in the ten's place, we shall have 15 tens. Then 15 tens divided into 11 equal parts will give 1 ten for each of these parts, and 4 tens over, undivided. If we put down this 1 ten in the ten's place, and convert these 4 tens into 40 units, and to them add the 4 units already in the unit's place, we shall have 44 remaining to be divided into 11 equal parts. Each of these parts, it is plain, will be equal to 4 units, which we set down, and we have 2 hundreds 1 ten and 4 units for each of the parts, if we divide 2 thousands 3 hundreds 5 tens and 4 units into 11 equal parts. If, therefore, we divide 2354 by 11, the quotient will be 214.

Example 3. Divide 1548 by 9.

Example 4. Divide 4235 by 6.

We will now propose an example in LONG DIVISION, in which you will recollect the Divisor exceeds 12. This species of Division is usually attended with great difficulty and obscurity to the pupil. By a little attention, however, you will be able, by the use of the solids, perfectly to understand it.

* The *full meaning* of the inquiry, How many times are 11 units contained in two thousands? or, what will *one* of the parts amount to if 2 thousands be divided into 11 equal parts? if properly examined will be sufficient to show, why it is indeed impossible to divide 2 thousands into 11 equal parts. I am here required not only to divide, literally, the 2 thousands into 11 equal parts, but am also required to say How many *thousands* there will be in each of these parts. The *entire sense* of the inquiry, then, is not merely this, viz. If I divide 2 thousand into 11 equal parts, what will one of these parts be? But it is this, If I divide 2 thousands into 11 equal parts, how *many thousands* are there in each of these parts? And as there are *no* thousands in each of these parts, we say, in this view, that *we cannot divide the* 2 *thousands into* 11 *equal parts.* Inasmuch, then, as there cannot be even *one* thousand in each of these parts, we take the 23 hundreds, or the 2 thousands and the 3 hundreds, and proceed to the question, If 23 hundreds are divided into 11 equal parts, how many *hundreds* are there in each of those parts, &c. We see, then, that the requisition is to say, not only *what*, in these divisions, each part amounts to, but to specify *how many of that* ORDER *of* UNITS, *which we are thus dividing, each part amounts to.*

Let it, then, be required to divide 232 by 22. In this example, we have to divide 2 hundreds 3 tens and 2 units into 22 *equal parts*, or, which is the same thing, to ascertain how many times 2 tens and 2 units *are contained* in 2 hundreds 3 tens and 2 units; or, which is still the same thing, to see how many times 2 tens and 2 units can be *taken from* 2 hundreds 3 tens and 2 units. We have the figures expressing this example, and the delineation of the solids illustrating it, in Plate XI. Figure 5. We will, however, consider the *erroneous* manner in which we proceed, and which you see exemplified, in the *first* collection of figures, which occurs in Figure 5, Plate XI. In this collection, which stands thus, 22)232(1
22

we first say, according to the Rule for Long Division laid down in the books, 22 in 2 goes *no* times; but 22 in 23 goes once; and we set down 1 in the quotient. We then, according to the same Rule, multiply the Divisor 22 by this 1, which is the first figure in the quotient, and say once 2 is 2; which 2 we have here set down under the first 2 in the Dividend, and then say once 2 is 2, setting down this last 2 under the 3 in the Dividend. Every pupil, who has committed to memory the Rule above mentioned, will say, that those two 2s under the Dividend are not correctly set down. Now I can *prove*, that these two 2s *are* correctly set down, although I know that they are *not* correctly set down; and not one pupil in a hundred will detect the sophistry of the demonstration. First, then, it will be seen, that *once* 2 *units* is 2 *units*, and that *once* 2 *tens* is 2 *tens*. But when I multiply the right hand 2, or the 2 units in the Divisor, by the 1 in the *Quotient*, I say *once* 2 *units;* and once 2 *units* makes 2 *units*, and this 2 units, since they are to be *subtracted* from the Dividend, must be placed under the 2 *units* in the Dividend. And again, once the 2 tens in the Divisor makes 2 tens, which I set down under the 3 tens in the Dividend, in compliance with the Rule for subtraction, which requires me to put *units under units, tens under tens, &c.* Have I not, then, *proved*, that the 22 is here set down correctly? It is no refutation of this proof, to say that we divide the 23 only in this step of the operation, and not the whole 232; for, after all, the fact is undeniable, that once 2 units are 2 units, and once 2 tens are 2 tens, and in order

to *subtract* them correctly from the Dividend, *we must put units under units, and tens under tens.** Now, if you attend to the illustration by the solids you will see wherein the fallacy of this reasoning consists; and you will also clearly understand the principles of what is called *Long Division.* By referring to the illustration by means of the solids, which you see delineated in Plate XI. Figure 5, you will perceive, that we have the 2 hundreds 3 tens and 2 units, to be divided into 22 equal parts. The 2 hundreds it is impossible to divide into 22 equal parts so as to express the amount of one of these parts by figures. So we must reduce these 2 hundreds to 20 tens, which added to the 3 tens in the 232 will make 23 tens. If we divide 23 tens into 22 equal parts, 1 ten will be equal to each of these parts, and there will be 1 ten over, undivided; and the 1 ten which is equal to one of these parts will represent the first figure of the *Quotient.* Now we will multiply the Divisor by this 1 in the Quotient. We say, therefore, 1 time 2 is 2; but 2 what? The answer is 2 tens, because we multiply 2 *units* by 1 *ten;* and units multiplied by tens, you know, produce tens. Again, 1 time 2 is 2; but 2 what? The answer is 2 hundreds, because I multiply the 2 tens of the Divisor by the 1 ten in the Quotient; and tens multiplied by tens produce hundreds. The 1, or the first figure in the Quotient, is 1 *ten,* as you have ascertained by using the solids in illustrating this example. The fact, that this one means 1 ten you never could have ascertained by the figures merely, until you had finished the example. But you must now be convinced, how necessary it is in Arithmetic to understand thoroughly, *as you proceed,* the precise *nature* and *reason* of every step. After having put our 22, or 2 hundreds and 2 tens under the 2 hundreds and 3 tens of the Dividend, we subtract them, in order to ascertain how many of the hundreds or tens, or of both, will remain over, undivided, after dividing 23 tens of the Dividend into 22 equal parts. We find by this Subtraction, that there is 1 ten over, to which we bring down the 2 units of the Dividend. We then inquire, How many units there will be in each of the parts, if we divide the 12 units or 1 ten and 2 units into 22 equal parts.

* Nor does it obviate the difficulty to say, that this 1 in the Quotient is in reality 1 *ten.* For this fact the pupil is not supposed to know, until the operation is finished.

The answer will, of course be *no* units; a cypher therefore, is placed in the *Quotient* to signify, that there are *no units* belonging to this *Quotient*.* On dividing 232 by 22, then, we have 10 for our Quotient and 12, or 1 ten and 2 units, for a remainder.

Example 2. Divide 5104 by 232.
Example 3. Divide 2334 by 37.
Example 4. Divide 3152 by 42.
Example 5. Divide 289 by 17.
Example 6. Divide 4096 by 64.

There is one case in Division, which frequently costs the scholar much perplexity, and which we shall now proceed to examine and illustrate. This is the case in which *cyphers are found at the right of the Divisor.* Of this case we shall give the following example. Divide 13134 by 1300. Here we have to find how often 1 thousand and 3 hundreds are contained in 1 ten thousand, 3 thousands, 1 hundred, 3 tens, and 4 units; or to divide 1 ten thousand, 3 thousands, 1 hundred, 3 tens, and 4 units, into 1300 equal parts. The operation is written in figures, and its illustration delineated in Plate XII. Here, as it would be inconvenient to divide 1 ten thousand into 1300 equal parts so as express the amount of one of these parts in the quotient, we take this 1 ten thousand and the 3 thousands, or, which is the same thing, the 13 thousands. Now if we divide 13 thousands into 1300 equal parts, there will be 1 ten in each of these parts. Or if we inquire how many times are 1 thousand and 3 hundreds contained in 1 ten thousand and 3 hundreds; the answer will be, 1 ten times. We put this 1, or 1 ten, in the quotient. Then multiplying the Divisor by this quotient of 1 ten, and subtracting the product from the 13 thousands, in order to ascertain, if there is anything over, after dividing the 13 thousands by the 13 hundreds, we find that there is nothing over, and bring down the 1 hundred in one dividend, in

* The way in which this putting cyphers in the Quotient is commonly stated, is apt to lead the pupil into an error. He is told, that if, after the subtraction is performed and the next figure of the Dividend brought down, the number so made up of the remainder and this next figure of the Dividend annexed, will not contain the Divisor, a cypher must be put in the Quotient. This statement is calculated to impart to the pupil the notion, that the *circumstance* that the remainder in question with the next figure of the Dividend annexed, will not contain the Divisor, is the *reason* why the cypher is placed in the Quotient.

which 1 hundred the 13 hundreds cannot be contained at all. We have, then for our quotient 1 ten and *no* units: and, for a remainder, beside the 1 hundred, 3 tens and 4 units; in all 134. Now we are directed in the Rule for this case of Division, in which there are cyphers at the right of the Divisor, "to reject these cyphers, and to divide by the *figures* in the Divisor, neglecting as many figures at the right hand of the Dividend, as we have rejected cyphers in the Divisor; until we have divided the rest of the Dividend by the figures in the Divisor, and then to bring down the figures of the Dividend so neglected; and these neglected figures of the Dividend, with the last remainder, if there be any, will constitute the entire remainder." To learn the *reason* of this direction, you will see on inspecting the solids, that you have 1 thousand and 3 hundreds, or 13 hundreds for your Divisor, and 1 ten thousand, 3 thousands, 1 hundred, 3 tens, and 4 units, or 131 hundreds, and 3 tens, and 4 units, or 34 units besides, for your Dividend. Now the simplest way of ascertaining the quotient in this Divisior would be to enquire, *how many times the 13 hundreds, or the Divisor, are contained in the 131 hundreds of the Dividend;* and not to inquire, *how many times are 1300 units contained in 13134 units.* Rejecting the cyphers in the Divisor then, is nothing more, than calling it by the more simple name of 13 hundreds, instead of 1300 units. And after having made the Divisor *hundreds* instead of *units,* we must *do the same* with the Dividend, or call the Dividend 131 hundreds and 34 units, which is done by neglecting the 34 units, that is, as many figures in the Dividend as we rejected cyphers in the Divisor, when we divide; which 34 units must of course, remain, after the divisor is performed, with such of the 131 hundreds as are not divided. And these undivided hundreds, and the 34 units, constitute, the *Remainder.* In this example, there is 1 hundred over or undivided, which, with the 34 units, makes the remainder 134.* On the division of 13134 by 1300, the

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* This principle of rejecting the cyphers at the right hand of the Divisor, may be well illustrated, by calling our 13134 units 13134 *cents.* And our 1300 units, 1300 *cents.* Now in this instance, we might inquire, how many times are 1300 cents contained in 13134 cents. But it would certainly be far more simple, as well as more convenient to ask how many times are 13 *dollars* contained in 131 dollars and 34 cents. We should thus find that 13 dollars are contained in 131 dollars, 10 times and 1 dollar over, which added to the 34 cents, gives 1 dollar 34 cents, or 134 cents for a remainder. It is on this

quotient is 10, and the remainder 134. You may now illustrate with the solids, and perform at the same time with the figures, the following examples.

Example 2. Divide 1256 by 200.
Example 3. Divide 5160 by 430.
Example 4. Divide 10896 by 110.
Example 5. Divide 25555 by 2400.
Example 6. Divide 119564 by 1220.
Example 7. Divide 35880 by 1560.
Example 8. Divide 162347 by 15000.

We have thus explained the principles of the *Simple* Rules of Arithmetic, viz: Numeration, Simple Addition, Simple Subtraction, Simple Multiplication and Simple Division. These are also sometimes called the Decimal Rules of Arithmetic, because when we operate with them, we numerate, carry, and borrow for *ten*. This term *Decimal*, is derived from the Latin word, *decem*, which means *ten*, and these Decimal rules are called *simple*, on account of the regular or *simple increase* of the different *orders of units* in them, which is always *ten*. The unit of the *second order*, or a *ten*, has *ten times* the value of a unit of the *first* order, or a *unit*; a unit of the *third* order, or a hundred, has *ten times* the value of a unit of the *second* order, or a ten; a unit of the *fourth* order, or a *thousand*, has *ten times* the value of a unit of the *third* order, or a hundred; and so on by the same increase throughout. But in what are called the *Compound* Rules of Arithmetic, the increase is *irregular*: thus, *ten* units make a ten, *ten* tens make a hundred, *ten* hundreds make a thousand, &c., while *four* farthings make a penny, *twelve* pence make a shilling, *twenty* shillings make a pound; and on account of this *irregularity* of increase in these different denominations, viz. farthings, pence, &c., those Rules of Arithmetic, which relate to the Addition, Subtraction, Multiplication, and Division of these *different denominations* of things; whether *pounds, shillings* and *pence*, or *feet, yards*, and *rods*, or *tons, pounds* and *ounces*, or any other kind of money, measure, or weight; are called *Compound Rules*. Units, tens,

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principle of rendering the Divisor and Dividend more simple in their expression, and more *manageable*, in their operation, that we are directed to reject the cyphers at the right of the Divisor, and the same number of figures at the right of the Dividend, until the Divison is performed, by means of the figures in the Divisor.

hundreds, thousands, ten thousands, and hundred thousands, are called *Simple Numbers;* and farthings, pence, shillings, pounds, as well as all the other denominations of currencies, weights and measures, are called *Compound Numbers.* When we *put together* the *Simple Numbers,* we employ *Simple Addition or Simple Multiplication;* when we *take apart* the *Simple Numbers,* we employ *Simple Subtraction or Simple Division.* In the same manner, when we put together the *Compound Numbers,* we employ *Compound Addition,* or *Compound Multiplication;* when we take apart the *Compound Numbers,* we employ *Compound Subtraction* or *Compound Division.* The only difference, then, between the *Simple* and *Compound Rules* is this; that, when we operate with the *Simple* rules, we *carry and borrow one for every ten,* but when we operate in the *Compound Rules,* we carry and borrow *one for as many as it takes of a lower denomination to make one of the next higher.* For instance, if we add 4 shillings and 6 pence, to 3 shillings and 8 pence, we set them down in this manner

s.	*d.*
4	6
3	8
8	2

putting an *s* over the shillings to signify *shillings,* and a *d** over the pence to signify pence. We then say, 8 and 6 makes 14 pence or 1 shilling and 2 pence; we set down the 2 pence under the pence, and carry the 1 shilling to the shilling's place. We then say, 3 shillings and 4 shillings make 7 shillings, and 1 shilling we carry from the place of the pence makes 8 shillings. We thus find the *sum* of 4 shillings and 6 pence, when added to 3 shillings and 8 pence, to be 8 shillings and 2 pence. Here we carry one for 12, because it takes 12 of the lower denomination, which, in this instance, is the denomination of pence, to make 1 of the next higher denomination, or the denomination of

* D is the first letter of the Latin word *Denarius,* which signifies a *Penny.*

shillings. If we wish to ascertain the amount of 4 times 3 pence and 2 farthings, We place the figures in this manner,

	d.	qurs.*
	3 .	2
		4
s.		
1 .	2 .	0

and say, four times 2 are 8; as there are 2 pence in these 8 farthings and *nothing* over, we put a cypher in the farthing's place in the *product,* and carry the 2 pence to the place of pence, and say, 4 times 3 make 12 pence, and 2 pence we carry from the farthing's place, which make 14 pence. In these 14 pence there is one shilling, and 2 pence over. We set down the 2 pence in the place of pence, and carry the 1 shilling into a place *by itself* to the left of the pence, and set it down with an *s* over it, to denote its denomination of shillings. We obtain then, 1 shilling and 2 pence, by multiplying 3 pence and 2 farthings by 4. Here we carry, first, 1 for 4, then 1 for 12. That is, *one for as many of the lower denomination as, in each instance, make one of the next higher.* If we wish to *subtract* 2 shillings and 7 pence from 3 shillings and 6 pence, we place the figures in this manner

s.	d.
3 .	6
2 .	7
	11

Here we say, 7 from 6 we cannot take; but we may take one of the 3 shillings, and suppose it to be converted into 12 pence, which added to the 6 pence, in the place of pence, will make 18 pence. And 7 pence from 18 pence will leave 11 pence. We then say, 2 shillings from 2 shillings leaves *no* shillings: for the 3 shillings in the minuend, had 1 shilling taken away, to be converted into 12 pence. Or, if we merely *imagine* this 1 shilling to be taken away, the figure 3 in the minuend remaining the same, we say 1 *to carry* to 2 is 3 and this 3 taken from 3 leaves nothing, the result will be the same. We have then 11 pence, for the remainder, on subtracting 2 shillings and 7 pence

* Qurs. for quarters: because 1 farthing is the *fourth,* or *quarter part,* of a penny.

from 3 shillings and 6 pence. In this instance we *borrow and carry* 1, for as many of the lower denomination as take to make one of the next higher; that is, we borrow 1 from the 3 shillings and convert it into 12, or as many of the pence as take to make one of the shillings; and to increase the subtrahend equally, we carry 1 to the next higher denomination of shillings, by *adding* 1 to the 2 shillings in the subtrahend, for the 12 of the lower denomination of pence, which 12 makes one of the next higher denomination of shillings. If we would *divide* 2 shillings and 7 pence into 2 equal parts, we set down the figures in this manner,

$$\begin{array}{rrrl} & s. & d. & \\ 2) & 2\ . & 7 & \\ \hline & 1\ . & 3\ . & 2 \end{array} \quad qrs.$$

and say, if 2 shillings be divided into 2 equal parts, one of those parts will be 1 shilling. We set down the 1 shilling in the quotient, and then say, if we divide 7 pence into 2 equal parts there will be 3 pence in each of those parts, and one penny over, undivided. We set down the 3 pence, and change our 1 penny over, to 4 farthings, and dividing these 4 farthings into 2 equal parts, find that there will be 2 farthings in each of these parts. We set down our 2 farthings, at a short distance to the right of the 3 pence in the quotient, and thus find that the whole quotient arising from the division of 2 shillings and 7 pence, into 2 equal parts, is 1 shilling, 3 pence, and 2 farthings. In Division, then, we regard always the number of the lower denomination, which makes 1 of the next higher. We will now illustrate a few examples with the solids, and you will find little difficulty in acquiring a correct knowledge of the principles of the *Compound Rules*. You will see clearly, that the only difference between *them* and the *Simple Rules*, consists in the variation of the numbers, for which you *borrow* and *carry*.

COMPOUND ADDITION.

Add 1 pound 2 shillings and 6 pence to 1 shilling and 8 pence. See Plate XIII. Figure 1. Here we say, 8 pence and 6 pence are 14 pence or 1 shilling and 2 pence. We set down the 2 in the place of pence, and carry the 1 shilling to the shilling's place.

We then say, 1 shilling and 2 shillings make 3 shillings, and the 1 shilling we *carry* from the place of pence, makes 4 shillings. We set down these 4 shillings in the shilling's place, and, lastly, set down the one pound.* The sum, therefore, is £1. 4*s*. 2*d*.

		£.	s.	d.	qurs.		s.	d.	qurs.
Example 2.	Add	1	3	4	7	to	4	8	1.
Example 3.	Add		4	6		to		3	2.
Example 4.	Add		1	2	3	to		9	1.

COMPOUND MULTIPLICATION.

Now *multiply* 1 shilling, 4 pence, and 2 farthings by 3. See Plate XIII. Figure 2. We have here, placed the solids of shillings, pence, and farthings, and put our multiplier under the farthings, as the most convenient place. This multiplier, is 3 *units;* because we are required to take the multiplicand 1 shilling, 4 pence, and 2 farthings, *three* times, or 3 *units*, or 3 *ones* of a time. We say, 3 times 2 farthings are 6 farthings. In 6 farthings there is 1 penny, and 2 farthings over. We set down the 2 farthings, and carry the 1 penny to the place of pence. We then say, 3 times 4 pence are 12 pence, and 1 penny we carry from the place of pence, makes 13 pence. In 13 pence there is 1 shilling, and 1 penny over. We set down the 1 penny, and carry the 1 shilling to the place of shillings. We lastly, say, 3 times 1 shilling make 3 shillings, and 1 shilling we carry from the place of pence, will make 4 shillings. The product of 1 shilling 4 pence and 2 farthings multiplied by 3 is, therefore, 4 shillings, 1 penny, and 2 farthings.

		s.	d.	qurs.		
Example 2.	Multiply	3	6	3	by	5.
Example 3.	Multiply			3	by	10.
Example 4.	Multiply	3	3	3	by	3.
Example 5.	Multiply	5			by	4.

* The letter L with two marks across it, thus £, is put for *pounds*, from the circumstance, that L is the initial of the Latin word *Libra*, which signifies a pound; and the two marks, = are put across it to make it of the plural number; thus, £, or *Libræ*, *pounds*.

COMPOUND SUBTRACTION.

Subtract 1 shilling 4 pence and 3 farthings from 1 shilling 5 pence. See Plate XIV. Figure 1. In this example, we say, 3 farthings from *no* farthings, we cannot take; but if we take 1 of the 5 pence and change it into 4 farthings, and put these 4 farthings in the farthing's place, the *minuend* would be 1 shilling 4 pence and 4 farthings, which is the same thing as 1 shilling, 5 pence and *no* farthings. We then say, 3 farthings taken from 4 farthings leave 1 farthing. Then 4 pence taken from 4 pence, leave *no* pence, and 1 shilling taken from 1 shilling, leave *no* shillings. If, then, we subtract 1 shilling 4 pence and 3 farthings, from 1 shilling and 5 pence, the remainder will be 1 farthing.

		£.	s.	d.	qurs.		£.	s.	d.	qurs.
Example 2.	Subtract	1 .	1 .	1 .	1	from	1 .	1 .	1 .	2.
Example 3.	Subtract			2 .	3	from			3.	
Example 4.	Subtract				1	from	1.			

COMPOUND DIVISION.

Divide 1 shilling and 2 pence, by 4. See Plate XIV. Figure 2. The divisor, as you see, is 4 *units;* because we are required to divide the dividend, in this example, into 4 *units*, or into 4 *ones*, of parts. We therefore, say, that, since 1 shilling cannot be divided into 4 equal parts, or in other words, *since if we divide* 1 *shilling into* 4 *equal parts, there will be no shillings in each of these parts,** the 1 shilling must be converted into 12 pence, and added to the 2 pence, in the place of pence, which will make 14 pence. If we divide these 14 pence into 4 equal parts, there will be 3 pence in each of these parts, and 2 pence over, undivided. Now since, if we divide these 2 pence into 4 equal parts, there will be *no* pence in each of these parts, we

* In the same manner, when in *Simple Division*, we divide 232 into 22 equal parts, we say, that the 2 hundreds cannot be divided into 22 equal parts. Now we do not mean by this assertion, that it is *literally* impossible to divide 2 hundreds into 22 equal parts; but only, that, if we should divide 2 hundreds into 22 equal parts, there would not be so much as *one* hundred in each of those parts. We therefore, change the 2 hundreds to 20 tens, &c.

must change the 2 pence into 8 farthings, and we divide 8 farthings into 4 equal parts, there will be 2 farthings in each of these parts. The quotient, therefore, if we divide 1 shilling and 2 pence by 4, will be 3 pence and 2 farthings.

		£.	s.	d.	qurs.		
Example 2.	Divide	1	5	1	0	by	2.
Example 3.	Divide		3	6	3	by	3.
Example 4.	Divide	1	0	0	0	by	4.

It sometimes becomes necessary, in Arithmetical operations, to change money, weights, or measures, from one name or denomination to another. Thus we occasionally wish, for instance, to convert shillings into pence, shillings and pence into farthings, farthings into pounds, quarts into gallons, seconds into hours, &c. This changing of one denomination to another is called

REDUCTION.

If we wish to change a *lower* denomination into a *higher*, as farthings to shillings, the process is called

REDUCTION ASCENDING,

because we *go up* from a *lower* denomination to a *higher*. If, on the other hand, we are required to change a *higher* denomination to a *lower*, as pence to farthings, we call the operation

REDUCTION DESCENDING,

because we *go down* from a *higher* denomination to a *lower*. We shall give some examples, to show the *principle* on which these Reductions are performed.

Example 1. Reduce 1 shilling and 2 pence to pence. See Plate XIV. Figure 3. In order to reduce the shilling in this example to pence, we multiply it by 1 ten and 2 units, or by 12, because there are 12 times one penny in a shilling. We then put down the 12 pence, and add to them the 2 pence, which we find in the place of pence, in the number to be reduced, and it will make 14 pence for the number of pence in 1 shilling and 2 pence.

Example 2. Reduce 7 farthings to pence. See Plate XV. Figure 1. In this example, we divide the 7 farthings by 4 units, because there are 4 farthings in 1 penny; or, because there are one fourth part as many pence in 7 farthings, as there are farthings. Consequently, if we divide the 7, or the number of farthings, into 4 equal parts, one of these parts will be the number of pence contained in these 7 farthings. We find, then, that if, as in Plate XV. Figure 1, 7 farthings be divided by 4 units, the result will be 1 penny and 3 farthings.

		£. s. d. qurs.	
Example 3.	Reduce	1 . 2 . 5 . 0	to farthings.
Example 4.	Reduce	5 . 7 . 6 . 0	to pence.
Example 5.	Reduce	960	to pounds.
Example 6.	Reduce	250	to pounds.

VULGAR FRACTIONS.

The *Visible Numerator* is used in the illustration of Vulgar Fractions, by supposing different solids to be divided into the various other solids of a smaller size, and by substituting the solids of one size for those of another, according to the operations described in the Rules of the books. If we call the solid of a hundred thousand, *one*, or a *unit*, the solid of ten thousand will be *one tenth;* the solid of a shilling will be *one twentieth;* the solid of a thousand will be *one hundredth;* the solid of a hundred will be *one thousandth;* the solid of a ten will be *one tenth of a thousandth;* and the solid of a unit will be *one hundredth of a thousandth.* If we make the solid of ten thousand, *one*, the solid of a shilling will be *one half;* the solid of a thousand will be *one tenth;* the solid of a hundred will be *one hundredth;* the solid of a ten will be *one thousandth;* and the solid of a unit will be *one tenth of a thousandth.* If we make the solid of a thousand, *one*, the solid of a hundred will be *one tenth;* the solid of a ten will be *one hundredth;* and the solid of a unit will be *one thousandth.* If we make the solid of a hundred, *one*, the solid of ten will be *one tenth;* and the solid of a unit will be *one hundredth.* If we make the solid of ten, *one*, the solid of a unit will be *one tenth.* If we make the solid of a shilling, *one*, the solid of a thousand

will be *one fifth,* the solid of a penny will be *one twelfth,* and the solid of a farthing will be *one forty eighth.* If we make the solid of a penny, *one,* the solid of a farthing will be *one fourth.* To give a clear notion of the *numerator* and the *denominator,* or the two parts of which a Vulgar Fraction consists, take a solid of a shilling, as the one or unit, and suppose it to be divided into 5 equal parts, one of the solids of a thousand will represent one of these fifth parts. Now the figure 5 denotes the number of parts into which the *unit* is divided, and is called the *denominator,* and, for example, 3 of these fifth parts would be three fifths; and this figure 3 would be called the *numerator.* The numerator and denominator are written thus, $\frac{3}{5}$, and read *three fifths.* The numerator in a Vulgar Fraction, then, is written over the denominator, with a line drawn between them. The *denominator* shows, *how many parts the unit is divided into;* and the *numerator* signifies, *how many of these parts are contained in the fraction.*

We shall illustrate the *Addition, Multiplication, Subtraction,* and *Division* of Vulgar Fractions.

ADDITION OF VULGAR FRACTIONS.

1. Add $\frac{1}{10}$ to $\frac{1}{5}$. Let a solid of ten thousand represent the unit; then, a solid of a thousand will be $\frac{1}{10}$. Now we cannot add this $\frac{1}{10}$ to $\frac{1}{5}$ in its present condition; because the sum would be neither 2 *tenths,* nor 2 *fifths.** But you may see, on examintion, that two of the solids of a thousand, are $\frac{1}{5}$ of the solid of ten thousand. But two of the solids of a thousand are $\frac{2}{10}$ of your unit, which is the solid of *ten thousand.* $\frac{2}{10}$ and $\frac{1}{5}$ of the unit, therefore, are the same thing. We may, then, add $\frac{2}{10}$ and $\frac{1}{10}$ together, and the sum will be $\frac{3}{10}$. The Addition of Vulgar Fractions is performed, then, by *reducing them to the same denomination, and adding the numerators.*

* Thus if we add 1 shilling to 2 pence, it will make neither 3 shillings nor 3 pence. But reduce the shilling to pence, and we shall have 12 pence, which added to 2 pence makes 14 pence. We must, then, reduce them to the same *denomination,* in order to add them; and so we must reduce fractions to the same *denominator,* before we can add them together.

MULTIPLICATION OF VULGAR FRACTIONS.

2. ***Multiply*** $\frac{1}{12}$ by $\frac{3}{4}$. From the meaning of the word *multiply*, we are required in this example, to find what $\frac{3}{4}$ of $\frac{1}{12}$ amounts to. For this purpose we are very properly directed by the books to multiply the numerators together for a new numerator, and the denominators together for a new denominator; thus $\frac{1}{12}\times\frac{3}{4}=\frac{3}{48}=\frac{1}{16}$; that is, 3 times 1 equals 3, and 4 times 12 equals 48. And the product of $\frac{1}{12}$ by $\frac{3}{4}$ is $\frac{3}{48}$ which equals $\frac{1}{16}$. Now let us illustrate the propriety of this operation by means of the solids. Take the solid of a shilling for a unit, and one of the solids of a penny will be the $\frac{1}{12}$; and *one* of the farthing solids will be $\frac{1}{4}$ of this solid of a penny, or $\frac{1}{4}$ of $\frac{1}{12}$; *two* of the farthing solids will be $\frac{2}{4}$ of $\frac{1}{12}$; and 3 of the farthing solids will be $\frac{3}{4}$ of $\frac{1}{12}$. Now the assertion is, that $\frac{3}{4}$ of $\frac{1}{12}$ is equal to $\frac{3}{48}$, which equals $\frac{1}{16}$; that is, if we take 48 of the farthing solids, or 48 of our *one fourths of* $\frac{1}{12}$, they will equal the shilling solid, or one *unit;* and 3 of the same farthing solids, or $\frac{3}{4}$ taken 16 times, will amount to the same thing, as taking one of them 48 times, because 16 times 3 makes 48; or the product of $\frac{1}{12}$ multiplied by $\frac{3}{4}$ is $\frac{3}{48}$, or, which is the same thing, $\frac{1}{16}$.

SUBTRACTION OF VULGAR FRACTIONS.

3. *Subtract* $\frac{1}{2}$ from $\frac{7}{10}$. Take the solid of ten thousand for the unit. Then a solid of a shilling will be $\frac{1}{2}$, and a solid of a thousand will be $\frac{1}{10}$, and 7 of these thousand solids will be $\frac{7}{10}$. We may reduce the $\frac{1}{2}$ to the same denominator with the $\frac{1}{10}$, by taking 5 of the thousand solids, which are equal to the shilling solid, which is $\frac{1}{2}$ or $\frac{5}{10}$ of our *unit*. If, then, we take 5 of the thousands from 7 of the thousands, we have left for the remainder 2 of the solids of thousands, or $\frac{2}{10}$, on the subtraction of $\frac{1}{2}$ from $\frac{7}{10}$.

DIVISION OF VULGAR FRACTIONS.

4. Divide $\frac{3}{5}$ of $\frac{1}{20}$ by $\frac{1}{100}$. In the illustration of this example, let the solid of a hundred thousand be the *unit*. In the first

place, we will ascertain, what the $\frac{3}{5}$ of $\frac{1}{20}$ are equal to:* and by multiplying the numerators in these two fractions, for a new numerator, and the denominators for a new denominator we have $\frac{1}{20}\times\frac{3}{5}=\frac{3}{100}$. That is, if we take one of the shilling solids for the 20th part of our unit, $\frac{3}{5}$ of this $\frac{1}{20}$ will be 3 of the hundred solids, which are in fact, $\frac{3}{100}$ of one unit, viz. the solid of a hundred thousand. We have then, $\frac{3}{100}$ to be divided by $\frac{1}{100}$: now one of the hundred solids is evidently contained in 3 of the hundred solids 3 times; which confirms the direction of the Rule, which tells you, *that in dividing Vulgar Fractions by each other, you must invert the divisor, and proceed as in multiplication.* Thus, $\frac{3}{100}\times\frac{100}{1}=\frac{300}{100}$, or 300 hundredths, which are the same thing as three solids of a hundred thousand. For there are 100 of the solids of a hundred in the solid of a hundred thousand, and 300 of these solids must, therefore, be equal to 3 solids of a hundred thousand; so that, if we divide $\frac{3}{5}$ of $\frac{1}{20}$ by $\frac{1}{100}$, the quotient will be 3.†

DECIMAL FRACTIONS.

That class of Fractions, which we have already considered, are called *Vulgar Fractions,* in consequence of their *common use.* We now come to that class of Fractions, which on account of their regular diminution by *ten,* from a unit, are termed, *Decimal Fractions.* If, for instance, we divide our *one,* or *unit,* into *tenth* parts, *hundredth* parts, *thousandth* parts, &c., these fractions, *one tenth, one hundredth, one thousandth,* &c., are called decimals, because *one hundredth* is the *tenth* part of *one tenth, one thousandth* is the tenth part of *one hundredth,* &c.

In our illustrations of Decimal Fractions, we shall make the solid of *ten thousand,*‡ the *unit;* the *solid of a hundred thousand* will, of course, be the *ten,* the *solid of a thousand* will be the *tenth,* the *solid of a hundred* will be the *hundredth,* the *solid of ten* will be the *thousandth,* and the *solid of a unit* will be *the*

* This $\frac{3}{5}$ of $\frac{1}{20}$ is what is commonly called, a *Compound Fraction,* or a *Fraction* of a *Fraction.*

† The pupil will now be able, by the instructions already received in the use of the solids, to suggest and illustrate other examples in Vulgar Fractions, at his own discretion.

‡ By some inadvertence, the delineation of the solid of a shilling was erroneously made in Plate XV. Figures 2 and 3, instead of the delineation of the solid of ten thousand. The pupil, however, can easily rectify this mistake.

tenth of a thousandth. See Plate XV. Figure 2. The 6 marks, 11,1111, are written above the solids, with the *separatrix* between the 4th and 5th place from the right. Then beginning at this 5th place from the right, we say, *in numerating*, *units*, and then going to the left, *tens;* then, leaving the *unit's* place on the left, proceed towards the right, we say, *tenths, hundredths, thousandths, tenths of thousandths.* By dividing the solid of a unit into ten equal parts, we should have hundredths of thousandths, and one of these ten parts into ten equal parts, we should have millionths, and so on. We shall, however, be able to convey to your mind a perfect understanding of the principles of Decimal Fractions, if you strictly attend to the illustrations which we shall make, by means of the six solids delineated in Plate XV. Figure 2. You see then, that decimals increase and diminish in the same manner, in which the value of the places in the simple rules of whole numbers increase and diminish, so that, in adding, subtracting, multiplying and dividing Decimal Fractions, we "*carry and borrow*" for *ten.*

We shall, in the first place, teach you how to *write* and *read* Decimal Fractions. Example 1. Write in figures, *twenty-five hundredths.* Now if you were required to write twenty-five *bushels,* for instance, you would find no difficulty in so doing. You would simply set down 25; and to signify, that this 25 meant twenty-five *bushels,* would write the word *bushels* at the right hand of the 25; thus, 25 *bushels.* In the same manner, to write twenty-five hundredths, you set down 25. But you need some *word* or *mark,* to show, that this 25 means, 25 *hundredths.* Now you might, write *hundredths* at the right hand of the 25, thus, 25 *hundredths,* to signify, that the 25 meant twenty-five hundredths, just as you did the word *bushels* at the right hand of 25, to make the 25 signify twenty-five *bushels.* But there is a more convenient way than this, to make the 25 mean twenty-five hundredths. The 25 is, in the first place, written down precisely as if you intended to write twenty-five of anything else, as twenty-five *bushels,* twenty-five *apples,* or twenty-five *anything.* Now the only thing for you to ascertain is this, *what sign or mark you should make, in order to signify that your* 25 *means twenty-five* HUNDREDTHS, *and not* 25 SOMETHING ELSE. This sign or mark, which we write to signify *hundredths,* is simply a *comma,* (,). The question then is, *where* this comma should be *placed,*

in order to signify *hundredths.* Now this comma is always put, in Decimal Fractions, between the place of *tenths*, and the place of *units;* thus, 1,1, is read 1, or 1 *unit*, and 1 *tenth.* We must, therefore, ascertain in every instance, where this place of tenths will occur. If we numerate the places, as their values are illustrated by the solids in Plate XV. Figure 2, we say, in the first place, *units*, *tens*, for the whole numbers; then for the fractions, *tenths*, *hundredths*, *thousandths*, *tenths of thousandths;* or we may begin at the tenths of thousandths, and going from right to left, say, *tenths of thousandths*, *thousandths*, *hundredths*, *tenths*, *units*, *tens*, and we may also begin at the tenths of thousandths, and say, *units;* but this unit *is the unit of tenths of thousandths;* for the smallest solid in this instance, represents *one* tenth of a *thousandth*, or *one* or the *unit of a tenth of a thousandth.* After thus saying *unit*, meaning *the unit of the tenth of a thousandth*, we may call the next solid to the left *ten*, but it will be *ten* of *tenths of a thousandth;* the next solid will be a *hundred*, but a *hundred of tenths of a thousandth;* and the last solid on the left in the fraction, will be a *thousand*, but *a thousand of tenths of a thousandth.* So that we shall have, in the fractional part of the solids, as delineated in Plate XV. Figure 2, *one thousand* tenths of a thousandth, and *one hundred* tenths of a thousandth, and *ten* tenths of a thousandth, and *one* tenth of a thousandth, which make in all *one thousand one hundred and eleven tenths of thousandths*, thus written, ,1111; or, 1111 with the comma before the 1 at the left hand place, or before the 1 tenth; *which* 1 *tenth is the same thing as* 1 *thousand tenths of thousandth.* Now we may determine where the *comma* should be placed with regard to our 25, to signify *hundredths:* for the 5 is *units of hundredths*, and the 2 is *tens of hundredths*, and 1 *ten hundredths* is the same thing as *ten hundredths*, and *ten hundredths* are the same thing as 1 *tenth;** and 2 *ten hundredths*, are the same thing as 20 *hundredths;* which 20 hundredths, with 5 hundredths, make 25 hundredths; so you see *units* of *hundredths*, are the same thing as *hundredths*, and *tens of hundredths* are the same thing as *tenths.* Now we may numerate our 25, calling our 5 *units* of *hundredths*, and our 2 *tens* of *hundredths*; or thus, which amounts to the same

* Of this and similar facts, the pupil may be satisfied by measuring the solids, which denote the respective fractions, by each other.

thing; saying, *hundredths*, *tenths*, and then putting our *comma* before the *tenths*, our 25 hundredths will stand thus, ,25.

Example 2. Write in figures, the Decimal Fraction, *forty four thousandths*. Set down the forty-four, thus, 44, as if you wished to write forty-four *anything*. Your right hand 4 will be in the *unit's of thousandths place*, or, which is the same thing, in the *thousandth's place*. The other 4 will, of course, be in the *ten's of thousandths place*, or, which is the same thing, in the *hundredth's place*, and the next place to the *hundredth's place*, is the *tenth's place*. Now we must put a cypher in the tenth's place to signify, that there are *no* tenths, and then before the tenth's place, put the comma; and our *forty-four thousandths* will appear thus, ,044.

Example 3. Write in figures, *one tenth of a thousandth.* We write our 1, just as we should, to express one of *anything else*. Then say, *tenths of thousandths* for the 1, then *thousandths*, then *hundredths*, then *tenths*; and as there are *no* thousandths *no* hundredths and *no* tenths, we put cyphers in each of their places, to *signify no* thousandths, *no* hundredths, and *no tenths*, and then the comma, and our *one tenth of a thousandth* will be written thus, ,0001.

Example 4. Write in figures, *twenty-four millionths*. Write the 24; then beginning with the 4, say, *millionths;* then, for the 2, say *hundredths of thousandths;* then *tenths of thousandths*, then *thousandths*, then *hundredths*, then *tenths*. Since there are *no* tenths of thousandths, *no* thousandths, *no* hundredths, and *no* tenths, put cyphers in their places, to signify *no* tenths of thousandths, *no* thousandths, *no* hundredths, and *no* tenths, and after the *no tenths*, the *comma*. Your twenty-four millionths will then stand thus, ,000024.

Example 5. Write in figures, *two hundred and fifty-six tenths of thousandths*. Write your 256 as usual, then in *numerating*, say, for the 6, *tenths of thousandths;* for the 5, *thousandths;* for the 2, *hundredths;* then *tenths;* but because there are *no* tenths, you write a cypher, to signify *no* tenths, then the *comma*. The two hundred and fifty-six tenths of thousandths will then stand thus, ,0256.

The accuracy of this mode of writing Decimal Fractions depends, as you see, on the fact, that, (taking the last example for illustration,) the 6 *tenths of thousandths* are the same thing

as 6 *units* of tenths of thousandths; the 5 thousandths are the same thing as 5 *tens* of tenths of thousandths; the 2 hundredths are the same thing as 2 *hundreds* of tenths of thousands; and the no tenths are the same thing as the no *thousands* of tenths of thousands.*

Example 6. Write in figures, *two hundred and forty-four thousand in whole numbers,* and *annex one hundredth in decimals.* The figures will be 244000,01. You here write the whole number first, then the decimal. If you were required to write in figures, *two hundred and fifty thousandths,* the figures would be ,250. But since 25 *hundredths* are the same thing as 250 *thousandths,* you had better, in this, and all examples of the kind, in which, on writing them, there is a cypher at the right, obliterate this cypher at the right, for ,25 will express the same decimal as ,250.†

After these examples in *writing,* you will have little difficulty in *reading* Decimal Fractions.

Example 1. Read *in words* the decimal ,289. This collection of figures, 289, with the comma before it, is, of course, to be read, *two hundred and eighty-nine;* but two hundred and eighty-nine *what?* This is to be determined by *numerating* these figures beginning with the *tenths.* The 2 then is *tenths,* the 8 is *hundredths,* and the 9 is *thousandths.* The decimal ,289, therefore, is read *two hundred and eighty-nine* THOUSANDTHS. We read, then, the 289 just as we should two hundred and eighty-nine *anything else,* and we ascertain what to call it, by *numerating* as above.

Example 2. Read the decimal ,0034. This is 34 *something;*

* The subject of Decimal Fractions is rendered miserably obscure and perplexing in most of the Books on Arithmetic, by calling the *tenth* of a thousandth, *ten* thousandths, and by similar errors of phraseology in other parts of fractional numeration. *Ten* thousandths amount, evidently, to *one hundredth,* or to *a hundred times* as much as the *tenth* of a thousandth; to express which idea, the phrase *ten thousandths,* is used, after which the next place is called *hundred* thousandths, instead of the *hundredth* of a thousandth; whereas a hundred thousandths is manifestly equivalent to one tenth, which is *ten thousand times* as much as the hundredth of a thousandth: and it is, certainly, not very astonishing, that a learner of tender years, should be somewhat puzzled in *Decimal Fractions,* when the majority of our school-books on Arithmetic, are "*ten thousand leagues awry*" *at the very threshold of the subject.*

† The pupil may here be able to appreciate the amount of the formidable proposition, *that cyphers have no meaning, when written at the right hand of Decimal Fractions.*

and to determine what, we say, for the first cypher at the left, *tenths*, for the second cypher, *hundredths*, for the 3, *thousandths*, and for the 4, *tenths of thousandths*. The 34 is, therefore, thirty-four *tenths of thousandths*.

Example 3. Read the decimal ,0349789234. In this, and other instances, where the number of figures is large, we may ascertain how they should be read *as whole numbers*, by Numeration, beginning with the right hand figure. The 4 is *units;* the 3, *tens;* the 2, *hundreds;* the 9, *thousands*; the 8, *tens of thousands;* the 7, *hundreds of thousands*. Here we complete a *period* in numeration, and it will be well to point it off, by placing a comma at the *top*,* between the 7 and the 9. The decimal will then stand thus, ,0349'789234. The 9 of the second period will be *units of millions;* the 4, *tens of millions*, and the 3, *hundreds of millions*. The figures, then, in whole numbers would be read, *three hundred and forty-nine millions seven hundred and eighty-nine thousand two hundred and thirty-four*; and so they are to be read in *Decimal Fractions*. But three hundred and forty-nine millions seven hundred and eighty-nine thousand two hundred and thirty-four WHAT? To answer this question correctly, we begin at the tenth's place, or at the left hand, and numerate towards the right, thus, the 0 is *tenths;* the 3 is *hundredths;* the 4 is *thousandths;* the 9 is *tenths of thousandths;* the 7 is *hundredths of thousandths;* the 8 is *millionths;* the 9 is *tenths of millionths;* the 2 is *hundredths of millionths;* the 3 is *thousandths of millionths;* and the 4 is TENTHS OF THOUSANDTHS OF MILLIONTHS. The fraction, therefore, is read, *three hundred and forty-nine millions seven hundred and eighty-nine thousand two hundred and thirty-four* TENTHS OF THOUSANDTHS OF MILLIONTHS.

From the illustration afforded by these examples, the principles involved in the writing and reading of decimals, must be sufficiently obvious. The learner can multiply instances at pleasure.

We will now proceed to illustrate the *Addition*, *Multiplication*, *Subtraction* and *Division* of *Decimal Fractions*.

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* To prevent its being confounded with the comma, or the separatrix, of the decimal.

ADDITION OF DECIMAL FRACTIONS.

Add 1,25 to ,06. In this example, we have to add 1 *unit*, 2 tenths, and 5 hundredths, or one and twenty-five hundredths, to 6 hundredths. See Plate XV. Figure 3. We say 6 hundredths and 5 hundredths make 11 hundredths; set down 1 hundredth, and for the remaining 10 hundredths substitute 1 tenth, carrying it to the tenth's place. Then say 2 tenths, and 1 tenth we carry from the hundredth's place, make 3 tenths. We set down the 1 unit, and the sum of 1,25 and ,06 is thus found to be 1,31; that is, 1 unit, 3 tenths and 1 hundredth, or 1 and 31 hundredths.

Example 2. Add together ,005 and ,006.
Example 3. Add together 2,3456 and ,0005.
Example 4. Add together 1,009 and ,256.
Example 5. Add together ,309 and ,001.
Example 6. Add together ,0001 and 1,0009.

MULTIPLICATION OF DECIMAL FRACTIONS.

Multiply ,45 by ,5. We have, in this example, 4 tenths and 5 hundredths, or 45 hundredths, to be multiplied by 5 tenths. See Plate XVI. Figure 5. We say, in the first place, 5 times 5 make 25; but 25 *what?* To determine this, we must consider, that we multiply 5 *hundredths* by 5 *tenths*. Now if we multiply 1 hundredth by 1 tenth, that is, take ONE TENTH *of one hundredth*, the result or product will be 1 *thousandth; for* 1 *thousandth is the tenth part of a hundredth;* if we multiply 1 hundredth by 2 tenths the product will be 2 thousandths; if we multiply 2 hundredths by 2 tenths, the product will be 4 thousandths; and, universally, if we multiply *hundredths* by *tenths*, the product will be *thousandths*. Our 5 times 5, therefore, will be 25 *thousandths*, because we multiply *hundredths* by *tenths*. These 25 thousandths are the same thing as 2 hundredths and 5 thousandths. We put down our 5 thousandths in solids on the table, and carry the 2 hundredths to the hundredth's place. We then say 5 times 4 are 20; but 20 *what?* To answer this question, we must consider, that, if we multiply 1 tenth by 1 tenth, (that is, take 1 tenth one tenth of a time,) the product will be 1 hundredth; that, if we multiply 1 tenth by 2 tenths, the product

will be 2 hundredths; that, if we multiply 2 tenths by 2 tenths, the product will be 4 hundredths; and, that, universally, if we multiply *tenths* by *tenths*, the product will be *hundredths*. Our 5 times 4, therefore, produce 20 hundredths, because *tenths* multiplied by *tenths* produce *hundredths;* and, if to these 20 hundredths we carry the 2 hundredths from the thousandth's place, we shall have 22 hundredths, or 2 tenths and 2 hundredths, which we set down. We, then, have 2 tenths 2 hundredths and 5 thousandths, or 225 thousandths, for our product of ,45 multiplied by ,5. By using the figures, we say 5 times 5 are 25; set down the 5 and carry the 2, and say 5 times 4 are 20 and 2 we carry makes 22, and setting down the 22, we have the figures 225 in our product. Now we are directed in the *Rule* for the Multiplication of Decimal Fractions, "*to multiply precisely as in whole numbers, and point off as many figures from the right in the product for decimals, as there are decimal places both in the multiplier and multiplicand.*" In the present instance, we have *two* places of decimals in the multiplicand, and *one* in the multiplier, making *three* places of decimals in both. So that we point off *three figures* from the right, and our product will stand thus, ,225, which exactly corresponds with the illustration we have given with the solids.

Example 2. Multiply ,3 by ,2. See Plate XVI. Figure 2. We say here, twice 3 make 6; and the 6 is 6 *hundredths*, because we multiply *tenths* by *tenths*. But, if we say in the figures, twice 3 are 6, and set down our 6, we have only this *one figure* 6 in the product, and we are required by the rule to point off *two* figures in the product for decimals, because we have *two* places for decimals in the *multiplier* and *multiplicand.* To remove this difficulty, the Rule directs, that, if all the figures in the product are not so many as the decimal places in the multiplier and multiplicand, the deficiency must be supplied by prefixing cyphers. We therefore prefix *one* cypher to the 6, and then we can point off *as many places for decimals in the product as there are both in the multiplier and multiplicand.* Our product will then stand ,06, which corresponds with the illustration with the solids.

Example 3. Multiply 2,005 by ,35.

Example 4. Multiply 1,24 by 1,4.

SUBTRACTION OF DECIMAL FRACTIONS.

1. Subtract ,004 from ,233. Here we have 4 thousandths to take from 2 tenths 3 hundredths and 3 thousandths; or 4 thousandths from 2 hundred and 33 thousandths. See Plate XVI. Figure 1. We say 4 thousandths from 3 thousandths we cannot take; but, if we take 1 of the 3 hundredths and convert it into 10 thousandths, we shall have 2 tenths 2 hundredths and 13 thousandths. Now we may say 4 thousandths from 13 thousandths leave 9 thousandths; set down our 9 thousandths, and say *no* hundredths from 2 hundredths (for our *three* hundredths have become *two* hundredths) leave 2 hundredths. Setting down the 2 hundredths, we say no tenths from 2 tenths leave 2 tenths. Setting down the 2 tenths, we have, for the remainder of ,004 subtracted from ,233, 2 tenths 2 hundredths and 9 thousandths, or 2 hundred and 29 thousandths, or as expressed in figures ,229.

2. Subtract ,003 from ,2. Here we have to take 3 thousandths from 2 tenths. See Plate XVI. Figure 3. We say 3 thousandths from nothing we cannot take; but we convert 1 of the 2 tenths into 10 hundredths, leaving 1 tenth; then convert 1 of these 10 hundredths into 10 thousandths, leaving 9 hundredths, and then we have 1 tenth 9 hundredths and 10 thousandths, or, which is the same thing, *one hundred and ninety-ten thousandths*, equal to 2 *tenths*, or *two hundred thousandths.* Then we may say 3 thousandths from 10 thousandths leaves 7 thousandths, *no* hundredths from 9 hundredths leaves 9 hundredths, and *no* tenths from 1 tenth leave 1 tenth. We then have 1 tenth 9 hundredths and 7 thousandths, or 197 thousandths, for our remainder, or as expressed in figures, ,197, resulting from the subtraction of ,003 from ,2.

Example 3. From 2,0003 subtract ,1.
Example 4. From ,1 subtract ,0001.
Example 5. From 3,45 subtract ,0023.
Example 6. From 1, subtract ,1.

DIVISION OF DECIMAL FRACTIONS.

Divide ,225 by ,5. See Plate XVI. Figure 4. Now we cannot divide 2 tenths by 5 tenths, because 5 tenths are con-

tained in 2 tenths *no times*. If we divide the 22 hundredths in the dividend by 5 tenths the quotient will be 4; but 4 what? To answer this question correctly, we must consider, that, if we divide 1 hundredth by 1 tenth, the quotient will be 1 tenth; that is, 1 *tenth* is contained in 1 *hundredth* 1 *tenth of a time;* that, if we divide 4 *hundredths* by 2 *tenths*, the quotient will be 2 *tenths*, that is, 2 *tenths* are contained in 4 *hundredths* 2 *tenths of a time;* that, if we divide 6 *hundredths* by 2 *tenths*, the quotient will be 3 *tenths*, that is, 2 *tenths* are contained in 6 *hundredths* 3 *tenths* of a time; and, that, universally, if we divide *hundredths* by *tenths*, the quotient will be *tenths*. The quotient, therefore, if we divide 22 *hundredths* by 5 *tenths*, will be 4 *tenths*, and 2 *hundredths*, or 20 *thousandths*, over. If to these 20 *thousandths* we add the 5 *thousandths* in the dividend, we have 25 thousandths remaining of the dividend to be divided. If we divide 25 thousandths by 5 tenths, the quotient will be 5; but 5 what? To determine this, we must consider, that, if we divide 1 thousandth by 1 tenth, the quotient will be 1 hundredth; that is, 1 *tenth* is contained in 1 *thousandth*, 1 *hundredth of a time;* that, if we divide 4 *thousandths* by 2 *tenths*, the quotient will be 2 *hundredths;* that is, 2 *tenths* are contained in 4 *thousandths*, 2 *hundredths of a time;* that, if we divide 6 *thousandths* by 2 *tenths*, the quotient will be 3 *hundredths;* that is, 2 *tenths* are contained in 6 *thousandths* 3 *hundredths* of a time; and, that, universally, if *thousandths* be divided by *tenths*, the quotient will be *hundredths*. If 25 *thousandths*, then, be divided by 5 *tenths*, the quotient will be 5 *hundredths*. This 5 *hundredths* annexed to the 4 *tenths* already in the quotient, will make the whole quotient 4 *tenths* and 5 *hundredths* or 45 *hundredths*. Now we are directed by the Rule for the Division of Decimals, "*To operate with the figures precisely as in whole numbers, and then to point off from the right as many figures for decimals in the quotient, as the number of the decimal places in the dividend exceeds that of the decimal places in the divisor.*" In the present example, there are *three* decimal places in the *dividend* and *one* decimal place in the *divisor;* the number of decimal places in the dividend *exceeds* that of the decimal places in the divisor by *two*. We point off, therefore, the 5 and the 4 in the quotient for *decimal places*, and

the quotient resulting from the division of ,225 by ,5 will stand thus, ,45, which agrees with the result of the illustration by the solids.

Example 2. Divide ,125 by ,25.

Example 3. Divide ,245 by ,7.

Example 4. Divide ,4356 by ,66.

PROPORTION,

OR

THE RULE OF THREE.

The chief obstacle which occurs to the pupil in the study of the RULE OF THREE is, that he does not understand the reason *why*, when he multiplies the *third term* by the *second*, and divides by the *first*, the quotient of this division should be the *fourth* term of the Proportion. He is, indeed, aware of the fact, that, if according to the *Rule*, *after stating the question*, *he multiplies the third and second terms together*, *and divides the product by the first term*, he will literally fulfill the *prescription*, and perform the required operation. But the *reason* of this proceeding he does not comprehend. We will give one example, and the pupil will at once *see* the principle on which the operation *prescribed* in the *Rule* is based. If 10 dollars be the interest of 100 dollars, of how many dollars will 1000 be the interest. The answer will be 10000 dollars. To show the principle in question by this example, let the solid of *ten* represent the 10 dollars, the solid of a *hundred* the 100 dollars, the solid of a *thousand* the 1000 dollars; and the solid of *ten thousand** the 10000 dollars, or the *answer*. Place these solids in their proper order on your table, beginning with the solid of *ten*, and proceeding towards the right, as the *four terms* of *a proportion* are usually placed. See Plate XVII. Figure 1. The *first* of these solids bears the same proportion to the *second*, as the *third* does to the *fourth*; or the *first* is contained the same number of times in the *second*, as the *third* is in the *fourth*. Let us, then, see *how it is*, that multiplying the *third* by the *second*, and dividing by the *first*, will produce the

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* The learner will perceive, that the delineation of the solid of a shilling has been erroneously placed in the first Figure of Plate XVII. instead of the solid of ten thousand.

fourth. The first is 10, the second 100, the third 1000, the fourth 10000. If we multiply the third, which is a *thousand,* by the second, which is a *hundred,* the product will be a *hundred thousand,* and will be represented by the *solid of a hundred thousand,* a delineation of which you see in Plate XVII. Figure 1, at the right of the 10000 solid. Now, if you divide this *solid of a hundred thousand* into *ten equal parts,* or, in other words, divide it *by the first term of the proportion,* each of those parts will be *ten thousand,* which is the *fourth term.* This example illustrates the principles of what is called in the books the *Rule of Three Direct.* To explain the nature of the *Rule of Three Inverse,* we have only to put the solid of the *thousand* for the *first* term, the solid of *ten* for the *third* term, and the solid of a *hundred* for the *second* term. Now it is evident, that if we multiply the *thousand* by the *hundred,* that is, the *first* term by the *second,* the product will be 100000, as represented by the *hundred thousand solid.* If this solid of 100000 be divided into 10 *equal parts,* or, in other words *be divided by the third term,* one of those parts will be *ten thousand,* or the *fourth* term. This is precisely what takes place in the *Rule of Three Inverse.* The *first* term is multiplied by the *second,* and the product divided by the *third* term; and the quotient will be the *fourth* term. You may see the *Rule of Three Inverse,* as well as the *Rule of Three Direct,* illustrated in Plate XVII. Figure 1.

THE ROOTS.

The Roots have always presented a subject of great difficulty to the student. The explanations of them, as given in many of the books on Arithmetic, are very clear, if properly attended to; but the mind of the pupil is too often perplexed with a multitude of demonstrations, which, in their nature, imply a degree of mathematical knowledge, far beyond the attainments or the years of most young persons, to whom the subject of evolution, if illustrated in a proper manner, might be made perfectly intelligible. The difficulty seems not to be, to know what is meant by a *square* or a *cube.* It is entirely conceivable to almost any learner, that, if we wish to ascertain the length of the side of a square or the square *root* of a given number, we must find the number which

multiplied once into itself will amount to the superficial contents of said square, or produce the given number, and that, if we would know the length of one side of a cubical block, or the cube *root* of any number, we have only to obtain the number which multiplied *twice* into itself, will denote the solid contents of the block, or the cube root of the proposed number. These matters are exceedingly well explained in the Arithmetics of Adams and Smith, not to mention others of deservedly high reputation. But after all, there comes a *Rule* for extracting the *Square Root*, and a *Rule* for extracting the *Cube Root*, which my own observation of the intelligence of the scholars in some of our best seminaries on these subjects, has convinced me, are very imperfectly understood in *connection* with those principles of evolution, which are so well demonstrated in the treatises referred to. We will then take these *Rules* for the *extraction* of the Square and Cube *Roots*, *article* by *article*, and exhibit the *reasons* on which they are respectively founded.

THE SQUARE ROOT.

To obtain the square root of a number, we are directed in the first place, "*to point off said number, into periods of two figures each, by putting a dot over the unit's place, the hundred's place, and so on from the right hand to the left.*" To understand the *reason* of this *pointing off* the number, the root of which is required, into periods of *two figures* each, we must consider the nature of *Numeration*. We shall thus see, that the *squares* of *units*, will be *units*, for units multiplied by units produce units. If, for instance, we multiply 2 units by 2 units, *or find the square* of 2 units, the result will be 4 units; and if we multiply 9 units (the highest number of units*) into 9 units, the result will be eight one units; or, to speak in entire consistence with the *phraseology* of *Numeration*, 8 tens and 1 unit. Now the object of *pointing off* is, to indicate the number of figures which will be contained in the *root*. We see, that the *squares* of units will be units, or units and tens. Therefore, since finding the square

* We pursue the philosophy or *rationale* of our system of Notation, and mean, of course, *the units in the unit's place.*

root of a number, is merely the reverse of multiplying this number into itself, and thereby producing the square; if the number of which it is required to find the *square root*, be units, or units and tens, this square root will be units, that is, there will be just *one* figure in this root. So if we wish to obtain the square root of units, or units and tens, we place our point over the units to signify that there will be *one* figure in the root thus, $\dot{4}$, or $8\dot{1}$, and so of *any other number under one hundred.* Again, if we multiply one ten by itself, (and 1 ten is the next after 9 units in our system of Numeration,) its product or square will be one *hundred;* or if we multiply 9 tens and 9 units, or 99 by itself, (which is the highest number before we come to hundreds,) the product or square will be 9 thousands, 8 hundreds, *no* tens, and 1 unit, or 9801. And thus, if a number, of which the *square root* is to be extracted, be less than ten thousand, and more than 99, the root we know will consist of *two* figures. In this number 9801, (and in a similar manner for any other number over 99 and under 10000) we place a dot over the unit's place, and over the hundred's place thus, $9\dot{8}0\dot{1}$, to show, that the root will consist of *two* places, one for *tens* and the other for *units.* Once more; if we multiply 1 hundred by itself, (and 1 hundred is next to 99,) the product or *square,* will be 1 *ten thousands;* for *hundreds multiplied by hundreds produce tens of thousands;* or if we multiply 999 by itself, (which is the highest number before we come to thousands,) the product or *square* will be 99 tens of thousands, 8 thousands, *no* hundreds, *no* tens, and 1 unit, or 998001. Therefore if the number, of which we are required to find the square root, be over 9999, and less than 1 *million,* we know, that the root will consist of three places, viz. *hundreds, tens* and *units,* and we accordingly point off the number thus, $9\dot{9}8\dot{0}0\dot{1}$; putting a dot over every *second figure* beginning with units. We might thus go on, and prove by *squaring* 1 thousand and 9999, 10 thousand and 99999, that, from the nature of our system of Notation, which is merely the *language* of figures, it will invariably happen, that we must point off every *second* figure beginning with units, to denote the number of places, of which the root will consist.

If then, we take, for example, the number 22, and *square* it, this square will amount to 484. To illustrate this, let the square

in Plate XVII. Figure 2. A.,* be supposed to measure 22 feet on each side. The contents of this square will be 484 square feet, and are ascertained by multiplying 22 by itself. Now if we wish to find the length of one of these sides, when we know only what the square contents are, we *extract the square root* of 484. This number therefore, we point off into periods of *two* figures each, thus $\dot{4}8\dot{4}$. We thus see that the root, or one of the sides of our square, will consist of *two* places, *units* and *tens*. We are in the next place, directed by the rule, "*To find the greatest square in the left hand period, and to place the root of this square on the right hand of the number, whose root is to be extracted, like a quotient in Division.*" Now the greatest square in the left hand period, is, in this example 4 hundreds, produced by multiplying 2 tens by itself. These 2 tens we put to the right of 484 for the first figure of the root. We subtract the 4 hundreds, the square of this 2 tens from the left hand period, as we are further directed by the rule, and find that nothing remains. We have thus the square root of all our *hundreds* in the 484. If we take away these 400 square feet of the Figure A. in Plate XVII. Figure 2, we shall have 84 square feet remaining, of the whole superficial contents of the Figure A., or of 484 square feet. This remainder of 84 square feet, consists of two pieces of 20 feet long and 2 feet wide each, as Figure C. in the same Plate and Figure, and one piece measuring 2 feet by 2 as Figure D. in the same Plate and Figure. All these may be expressed in figures thus, $20\times2\times2=80$, and $2\times2=4$. The 80 feet added to the 4 feet equal 84 feet, which completes the square of 484. Now we are required by the *Rule*, after we have subtracted the square of the first figure of the root from the left hand period, "*To bring down the next period to the right hand of the remainder, for a dividend.*" In this case the remainder is nothing. Our dividend then, is 84, or the remainder of the square. We are here directed by the Rule, "*To double the figure of the root already found*, and *to place it at the left hand of the dividend for a divisor.*" In observing this direction, we merely write the length of both the side pieces of the whole square A. exclusive of

* The solid of a hundred may denote the Figure B. 20 feet by 20, the solid of a ten the Figure C., and the solid of a unit the Figure D., in Plate XVII. Figure 2.

the small piece at the upper right hand corner, which is 2 feet long and 2 feet wide. The whole length of both these side pieces will be 20+20=40 feet.* We are lastly directed by the Rule, "*To place such a figure at the right of the divisor, as multiplied into the divisor thus increased, will make a product equal to, or next less than the dividend.*" We, therefore, in compliance with this direction, annex 2 to the 40, or 4 tens, which make 42. For we have 84 as the superficial contents of the remainder of the square A, whose contents are 484 feet, after 400 feet, or a square equal to the figure B, are taken away. Now these 84 feet are contained in the 2 side pieces and the small square at the upper right hand corner of the Figure A. These two side pieces spread out into one figure, or *parallelogram*, would be represented by putting together twice the Figure C, and once the Figure D, as in Figure F. This Figure F, then, would be 42 feet long, and being 2 feet wide, would make out our 84 feet remaining of the whole square A; and if we multiply our divisor 42 into the 2 units, which is the next and last figure of the square root of 484, it will exactly make 84, the said remainder. So that, to make up the whole square A, of 484 feet, we have the following items. The square B, of 400 feet, the two side pieces each signified by the Figure C, of 20 feet long and 2 feet wide, making 80 feet, and the corner piece, denoted by D, 2 feet square —which makes 4 square feet. These items may be summed up briefly as follows.

Figure B=20×20=	400 feet.
Twice Figure C=20×2×2=	80
Figure D=2×2=	4
	484

of which we have found 2 tens and 2 units, or 22, to be the *square root*.†

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* For 2 tens or 20 is the first figure of the Root.

† In the illustration of the principle of the *Square Root*, as directed in a previous note, we use the solid of 1 hundred to represent the superficial contents of 400 feet; two of the ten solids for the two side pieces of 20 feet long, and 2 feet wide each to represent the 80 feet, and a unit solid for the corner piece of 2 feet by 2, or 4 square feet, which completes the square.

THE CUBE ROOT.

To find the *cube root* of any number, we are directed by the Rule, in the "first place, to point off the number into periods of *three figures* each, beginning with the unit's place." To know WHY *we point off* in this manner in the *Cube Root*, we must attend to the following considerations. If we multiply 9 units by itself *twice*, that is, if we cube 9 units, (which is the highest number of units next to 1 ten,) the product, or the cube, will be 7 hundreds 2 tens and 9 units, or 729. In all numbers less than 1 thousand, therefore, the cube root will consist of one figure. For, by finding the cube root of a number, we only mean to find such a number as, multiplied into itself *twice*, will exactly produce that number; and if the root consist of more than one figure, the cube will not be less than a thousand. So that, in all numbers which are less than a thousand, the cube root must necessarily consist of *one figure*. Now, to find the cube root of 729, we put a dot over the unit's place, thus, $72\dot{9}$, to show that the root will contain one figure, and so of all numbers less than 1 thousand. Again, the cube of 1 ten will be 1000, and the cube of 9 tens and 9 units, or 99, (the highest number next to hundreds,) is 9 hundred thousands 7 ten thousands *no* thousands 2 hundreds 9 tens and 9 units, or 970299. In all numbers, therefore, less than 1 million, and over 999, the cube roots will consist of *two* figures. If we are required then to find the cube root of 970299, we point it off thus, $97\dot{0}29\dot{9}$: beginning with the unit's place and putting a dot over every *third figure*, to denote that the root will consist of *two* figures, and thus of any number over 999 and less than 1,000000. If we find the cubes of 999, the next highest number to 1000, and 9999, the next highest number to 10000, we shall find, that to obtain the cube roots of these cubes respectively, we must point off said cubes into periods of three figures each, beginning with units, according to the Rule. If we would for example, find the cube root of the number 10648, we point off this number into periods of three figures each, thus $1\dot{0}64\dot{8}$. We are then directed by the Rule, "*To find the greatest cube in the left hand period, and place its root on the right hand of the number,*

whose root is to be extracted, like a quotient in Division, for the first figure of the root." The greatest cube in the left hand period of this example is 8, that is, 8 thousands, and the root of this 8 is 2, or *two tens*, for the cube roots of *thousands* are *tens*; since if we cube 1 ten, it makes 1 thousand. Let this cube of 8000 be represented by one of the solids of a *thousand*, supposing this solid to be 20 feet on a side. It will, of course, contain 8000 solid feet; for $20\times20\times20=8000$. We are then required by the Rule "*To subtract this cube from the first period, and bring down the next period for a* DIVIDEND." This subtraction of the 8 *thousands*, will leave 2648, of which we are required to find the cube root, in order to make up the whole cube root of 10648. Now this 8000 solid feet, which is thus taken away from the 10648 solid feet, may be represented by the solid of a *thousand*, as you see in Plate XVII. Figure 3, A, which we will suppose to be 20 feet long, 20 feet broad, and 20 feet thick; and $20\times20\times20=8000$, the solid contents of the block A, of which the 2, that is, the 2 *tens*, or 20 feet, forming the first figure of the cube root of the number 10648, is one side. We are next directed by the Rule, "*To multiply the* SQUARE *of this figure of the root by* 300, *and to call the product the* DIVISOR." This divisor is made up of the square contents of three solids of 20 feet square each, which must be added to the solid A, towards making up the whole cubic block D, in the same Plate and Figure, whose whole cubic contents are 10648 feet; and each side of which measures 22 feet. We represent each of these squares (which taken together are called in some books the *triple squares*) by a solid of a *hundred*. These three squares taken together, will amount to 1200 square feet: for $20\times20=400$, the square contents of one of them, and $400\times3=1200$, the square contents of the three. This square content of the three squares may be found quite conveniently, by squaring the 2 *tens* in the first figure of the root, and multiply the square by 300; thus $300\times2^2=1200$, (the square of 2 or 2^2 being 4,) which precisely conforms to the direction in the Rule. We are next required by the Rule "*To find how often the divisor is contained in the dividend, and to put the result in the root for the next figure of the root.*" To understand clearly the *reason* of this requisition of the Rule, we must consider, that the object in

dividing the 2648 by 1200, is *not* to find how many times the divisor is contained in the dividend,* but *to ascertain the thickness of each of these three squares*, † in order to make up an exact cube as in Figure D. Now, inasmuch as 2648 solid feet constitute the remaining number of solid feet, to make up the whole solidity of 10648 feet Figure D, and since there must be contained in this remainder of 2648 solid feet, the solid contents of these three squares at least, the 2648 will contain 1200 at least as many times, as there are feet in the thickness of each of these three squares. So we divide the 2648 by 1200 and find, that the *quotient*, or rather the *thickness* of each of these three squares, is 2 feet, which is the next and last figure of the root. On applying these squares to the original cube A, we shall obtain B. But we see that, to make up the whole cube D, there is a deficiency at the corners, which deficiency we will supply by putting in these corners three solids of *ten*, which we suppose to be 20 feet long and 2 feet square each, and we shall have, $202 \times 2 \times 3 = 240$ solid feet belonging to these three pieces, to be added to the 2400 solid feet, towards making up the whole cube D. This sum will be $2400 + 240 = 2640$ solid feet, which lacks only 8 solid feet of amounting to the solid content 2648 feet. And we accordingly find, that, when the three corner pieces of 80 solid feet each are added to the Figure B, and form the figure C, there is still a small space not filled up, seen in Figure C, at the upper right hand corner. This small space may be filled by one of the solids of a unit, which we suppose to be 2 feet long, 2 feet broad, and 2 feet thick, and which will thus make $2 \times 2 \times 2 = 8$ solid feet, and added to the 2640 solid feet, will make up the remainder of 2648 solid feet. If we put the solid of a unit into the upper right hand corner, therefore, of figure C, it will exactly make up

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* For sometimes it will be found in practice, that this *dividend* as it is called, will contain its *divisor* too *many times;* that is, the *quotient* arising from this division will be greater than the thickness of each of the three squares, which are added to make up the whole cube. In such a case, the excess of the *quotient* will be manifest, when we make up the whole cube, and the quotient figure must accordingly be reduced.

† The fact, that the "*quotient*" in question denotes the *thickness* of each of the side pieces, which are added to make up the whole cube, does not proceed, as is erroneously stated in the demonstrations of the *Cube Root*, commonly found in the books, from the principle, that, *if we divide the cubic, by the superficial contents of a body, the quotient will be the thickness.* For this principle, though correct in itself, has here no application, since our dividend is *not* the cubic contents of the three squares, but the *whole* remaining solidity of the cube. This division is founded entirely on the principle of *approximation*.

D, or a cube containing 10648 solid feet. Now we are directed in the Rule, after we have placed the result of the division of the *dividend* by the *divisor* in the root for the next figure of said root, "*To multiply the divisor by this next figure of the root, and to this product to add the product of the former figure or figures of the root by the product of* 30 *multiplied by the square of the last figure of the root; and to add to these products the cube of the last figure of the root; calling the sum the* SUBTRAHEND, *which is to be subtracted from the* DIVIDEND, *and so on until the whole is finished.*" Now we shall see, on a review of what we have done, *how* and *why* we comply with this direction of the Rule. *First*, then, the product of "*the divisor by this next figure of the root,*" is $1200 \times 2 = 2400$ solid feet, the solid contents of the three squares put on to Figure A, to make Figure B. *Secondly*, the number to be added to this product, viz. "*the former figure or figures of the root by the product of* 30 *multiplied by the square of the last figure of the root,*" is $2 \times 30 \times 2^2 = 240 =$ the solid contents of the three corner pieces of 20 feet long 2 feet broad and 2 feet thick each, which added to the Figure B, form the Figure C. For $20 \times 2 \times 2 \times 3 = 240$, that is, the product of 3 into 80, the solid content of one of these corner pieces, is the same thing as "*the product of the former figure* 2 *of the root by the product of* 30 *into* 4, *the square of* 2, *the last figure of the root.*" *Thirdly*, the last number to be added to this product of "*the divisor by the next figure of the root,*" is "*the cube of the last figure of the root,*" $2 \times 2 \times 2 = 8$. This cube is a solid of 8 solid feet added to the Figure C, to complete the whole cube D. The sum of all these products will then stand as follows:

$1200 \times 2 =$	"Product of the divisor by the next figure of the root."	2400
$2 \times 30 \times 2^2 =$	"Product of the former figure of the root by the product of 30 multiplied by the square of the last figure of the root."	240
$2 \times 2 \times 2 =$	"Cube of the last figure of the root."	8
	Subtrahend =	2648

This subtrahend we subtract from our dividend, and find that nothing remains; so that our work is finished, and 22 feet is found to be the length of one side of a cube containing $22 \times 22 \times 22 = 10648$ solid feet.

CONCLUSION.

If the pupil has duly attended to the preceding illustrations, he cannot have failed to gain an adequate idea of the science of Numbers, and a clear comprehension of the philosophy of the *Arabic Notation*, as the *medium* through which a knowledge of that science is usually conveyed to the mind.

The chief obstacle, indeed, which lies in the way of the learner, consists in the difficulty of his understanding the precise import of the *Language of Figures;* to surmount which obstacle, it is absolutely essential that he should *learn* this *language*.

As we have previously intimated, *number* is one of the earliest and simplest ideas which the human mind acquires. Why is it, then, that *number* is so intelligible, and that Arithmetic, which is the "Science of Number," is so difficult of acquisition? The answer is obvious. It is because the Doctrine of Notation, or the Language which is used in teaching Arithmetic, is imperfectly understood.

The pupil, then, has a *new language* to learn, which is as different from his vernacular tongue, as is the Hebrew, the Greek, or the Latin; and without a knowledge of this language, he can no more obtain an *adequate* idea of Arithmetic, as taught by means of it, than he can understand the harangue of a savage, pronounced in the unknown dialect of a Pequod or a Choctaw.

In learning the Language of Figures, the principal difficulty consists in a confused and an erroneous idea of the *value* of the *places*, or of the different *orders of units*, in Numeration. To remove this difficulty in the most effectual manner, we must have recourse to the *simplest suggestion*, by means of which the *idea* of *value* is imparted to the mind. And if we observe the mode in which children, at a very early period, obtain this idea of value, we must be led to the conclusion, *that the magnitude, or the size of objects, is the primary suggestion of their value.* The child infers, that an apple, for instance, which he supposes to be twice as large as another apple, is worth twice as much, or possesses twice the value. We well know, too, that a child, before he understands the real value of different coins, prefers a *cent*

to a fourpence-half-penny, because the cent is the "*biggest.*" With much more certainty would he make this inference, if, in addition to the superior size of the *cent*, these two coins were apparently homogeneous, or of the same color. This propensity to infer the *value* of things from their *size*, or visible dimensions, discovers itself not only in childhood, but in subsequent life; and thus indicates the most feasible means of suggesting to the mind of the beginner an intelligible notion of the relative value of numbers, in the different places of Notation.

The satisfactory clearness, then, with which the illustrations of this work are effected, arises mainly from taking advantage of this early and obvious principle of mental operation; and the ease and rapidity with which the pupil, by means of these illustrations, obtains ideas on the subject of Arithmetic, which he has in vain spent months and even years to acquire, may be accounted for in the simple fact, that, in the preceding Treatise, we have used the same means to impart the "SCIENCE OF NUMBER," which, from time immemorial, have been employed in teaching the "SCIENCE OF EXTENSION."

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